A Spectral Finite-Volume Method for the Shallow Water Equations

BYOUNG-JU CHOI
Institute of Marine and Coastal Sciences, Rutgers–The State University of New Jersey, New Brunswick, New Jersey

MOHAMED ISKANDARANI
Rosenstiel School of Marine and Atmospheric Science, University of Miami, Miami, Florida

JULIA LEVIN AND DALE B. HAIDVOGEL
Institute of Marine and Coastal Sciences, Rutgers University–The State University of New Jersey, New Brunswick, New Jersey

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ABSTRACT

A spectral finite-volume (SFV) method is proposed for the numerical solution of the shallow water equations. This is the first phase in the development of a layered (isopycnal) ocean model. Its target applications include, in particular, the simulation of the wind-driven oceanic circulation in geometrically complex basins where layer outcropping and/or isopycnal–bathymetry intersection must be handled explicitly. The present formulation is geometrically flexible and can extend accuracy to arbitrary high order with no change to the basic algorithm. A flux-corrected transport (FCT) algorithm ensures the stability of the computations in regions of vanishing layer thickness and in areas where the flow features are underresolved. The spatial discretization is based on a two-level grid: a globally unstructured elemental grid and a locally structured grid consisting of $N \times N$ quadrilateral cells within each element. The numerical solution is continuous within each element but discontinuous across elements; the discontinuity is resolved by upwinding along characteristics. The accuracy and convergence rate of the SFV method are verified on two linearized problems amenable to analytical solution; the SFV solution exhibits a convergence order of $N + 1$ for smooth solutions. The FCT portion of the model is tested by simulating the formation of an oblique hydraulic jump in a supercritical channel flow. The model is then applied to simulate, in reduced-gravity mode, the double-gyre and wind-driven upper-ocean circulations in a square basin. Finally, the previous experiment is repeated in the North Atlantic basin to illustrate the application of the model in a realistic geometry.

1. Introduction

This article describes the development and testing of a high-order, geometrically flexible, and "robust" finite-volume method to solve the shallow water equations. The long-term goal is the development of a three-dimensional, hydrostatic ocean model that relies on isopycnal (density) coordinates in the vertical direction. The present model is a first step toward that goal.

The choice of vertical coordinate system, and its associated discretization, is one of the central issues faced by ocean model developers. The three common choices, geopotential (z level), terrain-following, or isopycnal coordinates (Haidvogel and Beckmann 1999), are not optimal across all flow regimes in the ocean as demonstrated in several intermodel comparison experiments (Willebrand et al. 2001; Chassignet et al. 2000). The majority of existing finite-element (e.g., Lynch and Werner 1991; Iskandarani et al. 2003) and unstructured-grid finite-volume (Chen et al. 2003) ocean models belong to the class of terrain-following models. This choice largely reflects the desire to extend the models’ geometric accuracy in the vertical direction and to represent as faithfully as possible oceanic topography.

Our effort is aimed at developing an unstructured-grid-based model with an alternative, isopycnal, coordinate system in the vertical. The main applications targeted are idealized process-oriented problems enclosed within realistic basin geometries (e.g., the wind-driven circulation in the North Atlantic Ocean). The main advantages of an isopycnal representation of the vertical structure of oceanic flows are elimination of spurious diapycnal diffusion, reduction of pressure gradient errors near steep topography, efficient representation of baroclinic processes per vertical degrees of freedom, and finally ease of implementation since an isopycnal model
can be obtained by vertically stacking a set of shallow water models.

Isopycnal models have disadvantages. The major one among them, from a computational point of view, is the need to contend with vanishing layer thicknesses and isopycnals intersecting topography. Successful, finite-difference-based isopycnal models rely on some form of total variation diminishing (TVD) or positivity preserving schemes to cope with the above problems (Bleck and Boudra 1986; Hallberg and Rhines 1996). A similar approach is pursued herein to allow for layer outcropping to occur.

We have chosen the finite-volume (FV) method to build our present model because of its flexible grids and the availability of several “robust” slope/flux limiting procedures to enforce the TVD property. In addition, finite-volume methods possess the local conservation property, a very desirable feature during long integrations. The geometric flexibility in the present model derives directly from the unstructured nature of its computational grids. These grids are based on the decomposition of the flow domain into triangular or quadrilateral elements. Their size and orientation can be adjusted to fit the geometric and dynamical requirements of the simulation. Complicated basin geometry and local grid refinement in dynamically active regions can then be handled with ease.

Finite-volume formulations have traditionally been restricted to first- or second-order accuracy when applied to unstructured grids (Alcrudo and Garcia-Navarro 1993; Chen et al. 2003). A potential outcome is an excessive amount of numerical errors, largely dissipative but also dispersive in nature, that can only be controlled by a large increase in the number of finite volumes, and hence computational cost. The main limitation of these schemes is that their cells contain 1 degree of freedom per cell (the solution average), and hence a collection of cells is needed to fit a polynomial of degree higher than 1. This collection is complicated to assemble given the quasi-random arrangement of cells in an unstructured grid. The difficulty increases dramatically when designing higher-order schemes.

Giannakouros and Karniadakis (1992) and Giannakouros et al. (1994) proposed an alternative FV discretization scheme that builds on the hierarchy of grids available in the spectral element method. At the global level, the grid is unstructured and is composed of elements within which a locally structured grid is embedded. The spectral element machinery can then be used to construct a high-order polynomial covering the cells within an element. The method consists of a (structured) collection of finite-volume cells arranged into elements, such that the operations needed for high-order interpolation, reconstruction, integration, and differentiation are considerably reduced. Giannakouros et al. (1994) used quadrilateral elements for the spatial discretization. The finite-volume cell boundaries are defined by the Gauss–Lobatto roots of the Chebyshev polynomials. In addition, the authors incorporated a flux-corrected transport (FCT) algorithm (Zalesak 1979) to handle shock waves and strong discontinuities. The spectral element interpolation machinery provides the high-order fluxes while the low-order fluxes consist of piecewise-constant interpolation at the cell levels. This particular implementation of the spectral element–FCT method enforces the continuity of the function across elemental boundaries during the reconstruction step, which complicates its application to highly unstructured grids. Furthermore, the multidimensional version of the method reconstructs state variables by tensor product application of one-dimensional operators, and performance therefore degrades as the element shapes depart significantly from that of a rectangle.

Wang (2002) introduced a discontinuous high-order finite-volume method for the solution of conservation laws in one-space dimension, which he refers to as a spectral finite-volume (SFV) method. The SFV method has been extended to solve a two-dimensional scalar equation on a fully unstructured grid (Wang and Liu 2002). The two-dimensional elements are triangles that are further subdivided into polygonal control volume cells to support high-order data reconstructions. Several total-variance-diminishing and total-variance-bounded limiters have been adapted for the nonoscillatory capturing of discontinuities.

In this paper, we propose a new formulation for an SFV method based on quadrilateral elements. The quadrilateral shape of elements makes the choice of quadrilateral finite-volume cells natural. The solution in our approach is interpolated with a high-order polynomial within an element and is discontinuous across elements; characteristic information is used to resolve the discontinuity. An FCT algorithm is incorporated to enforce monotonicity near strong discontinuities or underresolved gradients and forms the basis for handling vanishing layer thickness in our shallow water simulations.

In section 2, we briefly describe the system of shallow water equations and present the formulation of the SFV
method. Accuracy and convergence rates are calculated for two simple problems amenable to analytic solution: the pure advection of a Gaussian hill in a rotating circular flow and a linear standing wave sloshing in a square basin. The FCT algorithm is reviewed in section 3 for the monotonic advection of mass and momentum, and the combined SFV–FCT scheme is tested for the problem of shock formation in a supercritical channel flow. In section 4, a reduced-gravity ocean model is described based on the SFV method. The model is then tested on two problems: double gyre, wind-driven circulation with layer outcropping and an upper-ocean current flow. In section 4, a reduced-gravity ocean model is described based on the SFV method. The model is then tested on two problems: double gyre, wind-driven circulation with layer outcropping and an upper-ocean circulation in a realistic North Atlantic Ocean. We conclude in section 5 with some remarks on future work.

2. Spectral finite-volume method

a. Shallow water equations

The shallow water equations (SWE) can be obtained by vertical integration of the 3D Navier–Stokes equations along with the assumptions of hydrostatic pressure and a vertically uniform horizontal velocity profile. This system of nonlinear equations representing mass and momentum conservation is primarily hyperbolic and may lead to discontinuous solutions. The SWE in vector and conservation form are

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S},
\]

where \( \mathbf{E} \) and \( \mathbf{G} \) represent the fluxes along the \( x \) and \( y \) directions, respectively. The vector of unknowns is

\[
\mathbf{U} = \begin{bmatrix} h \\ u h \\ v h \end{bmatrix},
\]

where \( h \) represents the layer thickness, and \( u \) and \( v \) stand for the depth-averaged velocity components along the \( x \) and \( y \) directions, respectively. The Cartesian components of the flux vectors and source terms are

\[
\mathbf{E} = \begin{bmatrix} u h \\ u^2 h + \frac{1}{2} g h^2 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} v h \\ u v h \\ v^2 h + \frac{1}{2} g h^2 \end{bmatrix},
\]

where

\[
\mathbf{S} = \begin{bmatrix} 0 \\ f v h + \frac{\tau_x}{\rho} - \kappa u h + \nabla \cdot (v h u) \\ - f u h + \frac{\tau_y}{\rho} - \kappa v h + \nabla \cdot (v h v) \end{bmatrix}.
\]

Here, \( g \) is the gravitational acceleration; \( f \) is the Coriolis parameter; \( \tau \) is the wind stress; \( \rho \) is the constant water density; \( \kappa \) is the bottom friction coefficient; and \( \nu \) is the horizontal viscosity.

b. Finite-volume formulation

We stress that the present model is not based on a variational form, but rather on a finite-volume formulation (Alcrudo and Garcia-Navarro 1993). Equation (1) is spatially integrated over a region \( \Omega \) to obtain the integral form of the equations:

\[
\int_0^1 \frac{\partial \mathbf{U}}{\partial t} \, d\Omega + \oint_\Gamma \mathbf{F} \cdot \mathbf{n} \, d\Gamma = \int_\Omega \mathbf{S} \, d\Omega,
\]

after application of the Gauss theorem. Here \( \mathbf{n} \) is the outward unit normal to the boundary \( \Gamma \) of \( \Omega \), and \( \mathbf{F} \cdot \mathbf{n} = E_{t_x} + G_{n_x} \). The first volume integral represents the time rate of change of the amount of \( \mathbf{U} \) in \( \Omega \), while the surface integral is the total flux of \( \mathbf{U} \) through the cell edges. Define \( \mathbf{U} \) as the average of \( \mathbf{U} \) over \( \Omega \), that is,

\[
\overline{\mathbf{U}} = \frac{1}{V} \int_\Omega \mathbf{U} \, d\Omega,
\]

where \( V \) is the area of \( \Omega \) in two dimensions and the volume in three dimensions. The finite-volume form of the SWE can now be written as

\[
\frac{d \overline{\mathbf{U}}}{dt} + \frac{1}{V} \oint_\Gamma \mathbf{F} \cdot \mathbf{n} \, d\Gamma = \overline{\mathbf{S}}.
\]

Equation (7) is exact; the numerical approximation comes from evaluating the boundary integral and in time stepping the area averages.

Equation (7) is an ordinary differential equation that must be integrated in time. We choose a third-order Runge–Kutta (RK3) scheme for the time integration as a good compromise between practicality and accuracy. The RK3 scheme is expressed as

\[
\begin{align*}
R(\overline{\mathbf{U}}, t) &= \text{ADV}(\mathbf{E}, \mathbf{G}) + \overline{\mathbf{S}}, \\
\text{ADV}(\mathbf{E}, \mathbf{G}) &= -\frac{1}{V} \oint_\Gamma \mathbf{F} \cdot \mathbf{n} \, d\Gamma, \\
R_0 &= R(\overline{\mathbf{U}}, t), \\
\overline{\mathbf{U}}_1 &= \overline{\mathbf{U}}(t) + \Delta t R_0, \\
R_1 &= R(\overline{\mathbf{U}}_1, t + \Delta t), \\
\overline{\mathbf{U}}_2 &= \overline{\mathbf{U}}(t) + \frac{\Delta t R_0 + R_1}{2}, \\
R_2 &= R(\overline{\mathbf{U}}_2, t + \frac{\Delta t}{2}), \\
\overline{\mathbf{U}}(t + \Delta t) &= \overline{\mathbf{U}}(t) + \frac{\Delta t}{6} R_0 + \frac{4R_2 + R_1}{6}.
\end{align*}
\]
Notice that the scheme evolves the cell averages in time while the fluxes and sources require the function values. An efficient and accurate machinery must then be devised to transform cell averages into function values (the reconstruction step), and vice versa (the averaging step). In the next section, we turn our attention to the issues of spatial discretization, reconstruction, and cell-average computation.

**c. High-order reconstruction**

The spatial discretization is based on a two-level grid in order to endow the method with geometrical flexibility and high-order accuracy. The first level consists of partitioning the flow domain into quadrilateral subregions, which we refer to in the following as elements. The elemental grid is allowed to be unstructured; the adjacency pattern can hence be varied according to geometric and accuracy requirements. Each element is subdivided into a structured grid of \( N \times N \) cells and is transformed by one-to-one mapping into a computational plane as shown in Fig. 1 (Karniadakis and Sherwin 1999). This local structured grid permits the use of high-order polynomials to interpolate, average, and reconstruct the solution; it also plays a role in formulating the FCT portion of the algorithm as will be seen in section 3. Each element then contains \( N^2 \) cells over which Eq. (7) will be solved.

In order to take advantage of the interpolation properties of spectral methods (Boyd 1994), we choose our interpolation points to be the Gauss–Legendre roots of Legendre polynomial of degree \( N \); \( L_N(\xi_k) = 0 \) with \( k = 1, \ldots, N \), where \( L_N(\xi) \) is the Legendre polynomial of degree \( N \). The cell edges are the \((N + 1)\) Gauss–Lobatto–Legendre roots of the same Legendre polynomial:

\[
(1 - \xi_k^2)L_N'(\xi_k) = 0, \quad k = 0, 1, \ldots, N. \tag{9}
\]

It is easy to show that the two sets of roots are staggered so that \( \xi_{k-1} < \xi_k < \xi_k \). The two-dimensional grid is obtained by tensor product combination of the one-dimensional grid. The cell \( \Omega_{kl} \) associated with the cell-averaged scalar variable \( \bar{U}_{ij} \) is defined by the region \([\xi_{k-1}, \xi_k] \times [\eta_{l-1}, \eta_l]\). A point value in our scheme is interpolated via a high-order Lagrangian interpolant:

\[
U(\xi, \eta) = \sum_{i=1}^N \sum_{j=1}^N U_{ij} h_i(\xi) h_j(\eta), \tag{10}
\]

where \( \xi \) and \( \eta \) are the coordinates in the computational plane, \( U_{ij} \) is the function value at the collocation point \((\xi_i, \eta_j)\), and \( h_i(\xi) \) are the Gauss–Legendre cardinal functions:

\[
h_i(\xi) = \frac{L_N(\xi)}{L_N'(\xi_i)(\xi - \xi_i)}, \quad i = 1, 2, \ldots, N. \tag{11}
\]

The cell averages \( U_{ij} \) can now be computed by integrating the Lagrangian interpolants over the area of each cell:

\[
\bar{U}_{ij} = \frac{1}{V_{ij}} \sum_{i=1}^N \sum_{j=1}^N U_{ij} \int_{\eta_{l-1}}^{\eta_l} \int_{\xi_{k-1}}^{\xi_k} h_i(\xi) h_j(\eta) |J| \, d\xi \, d\eta, \tag{12}
\]

where \( V_{ij} \) is the area of cell \((k, l)\) and \( J = (x_i y_j - x_j y_i) \) is the Jacobian of the mapping between physical space and computational space. The above operations can be cast as a matrix vector product that maps the \( N^2 \) values of \( U_{ij} \) into the \( N^2 \) cell averages \( \bar{U}_{ij} \):

\[
\bar{U}_{ij} = \sum_{k=1}^N \sum_{l=1}^N \mathbf{A}_{ijkl} U_{kl},
\]

\[
\mathbf{A}_{ijkl} = \frac{1}{V_{kl}} \int_{\eta_{l-1}}^{\eta_l} \int_{\xi_{k-1}}^{\xi_k} h_i(\xi) h_j(\eta) |J| \, d\xi \, d\eta. \tag{13}
\]

Unlike the scheme proposed in Sidilkover and Karniadakis (1993) and Giannakouros and Karniadakis (1994), the interpolation is discontinuous at element boundaries, and we make no attempt at enforcing continuity. This makes the scheme compact as the reconstruction is now local to each element. A further advantage of our scheme is that it is now possible to uniquely invert the mapping between the function values \( U_{ij} \) and the cell averages \( \bar{U}_{ij} \) since the averaging matrix \( \mathbf{A} \) is square. For the given \( \bar{U} \) vector of dimension \( N^2 \), we reconstruct the \( N^2 \) function values \( U = \mathbf{A}^{-1} \bar{U} \). The matrix \( \mathbf{A} \) can be computed and stored for each element as a preprocessing step.
Fig. 2. Structure of the HLLC approximate Riemann solver. Initially \((t = 0)\) there are two constant states on either side of the wall at \(x = 0\). Once the wall is removed, three waves (rarefaction wave with speed \(W_{L}\), contact discontinuity \(W_{C}\), and shock wave \(W_{R}\)) propagate away from the boundary. Solution consists of four constant states \((U_{L}, U_{R}, U_{R'}, \text{ and } U_{G})\) separated from each other by three wave speeds.

A high-order quadrature is used to evaluate the boundary integrals in Eq. (7). We adopt Gauss quadrature to evaluate these boundary integrals since it provides high accuracy and does not require flux evaluation at element corners (thus obviating the need to figure out the upwinding directions at these corners; Fig. 1). We use \(N\) Gauss quadrature points, which leads to an exact integration formula for polynomials of degree \(2N - 1\) or less:

\[
\int \mathbf{F} \cdot n \, d\Gamma = \sum_{j=1}^{N} w_{j} \mathbf{F} (x_{j}, y_{j}) \cdot n \Gamma, \tag{14}
\]

where \(w_{j}\) are the Gauss quadrature weights, \((x_{j}, y_{j})\) are the Gauss quadrature points, and \(\Gamma\) is length of a cell side. The interpolation of physical variables along cell sides is based on the interpolating polynomial \((10)\).

d. Interelemental conditions

The discontinuous representation complicates the flux computations near element boundaries where the solution is two valued. Upwinding along the characteristic direction is required to resolve this discontinuity and to calculate a unique value for the boundary flux; we use an approximate Riemann solver based on the Harten–Lax–van Leer contact (HLLC) flux as described by Toro (1999, 2001). We briefly describe this procedure in this section and refer the reader to the aforementioned references for more details.

The HLLC approach assumes that four constant states \((U_{L}, U_{R}, U_{G}, \text{ and } U_{G})\) are separated by three waves with speeds of \(W_{L}\), \(W_{G}\), and \(W_{R}\) for the smallest, middle, and largest signal velocities in the solution of the Riemann problem in Fig. 2. Subscripts \(L\) and \(R\) refer to left- and right-going waves emanating from the elemental boundary, and \(*\) refers to the middle wave. For the homogeneous shallow water equations (i.e., \(S = 0\)), integration of the conservation laws \((5)\) over an appropriate control volume \([x_{L}, x_{R}] \times [0, T]\) gives the HLLC numerical flux as

\[
\mathbf{F}_{x=0} = \begin{cases} 
\mathbf{F}_{L} & \text{if } 0 \leq W_{L}, \\
\mathbf{F}_{W_{L}} & \text{if } W_{L} \leq 0 \leq W_{*}, \\
\mathbf{F}_{W_{R}} & \text{if } W_{*} \leq 0 \leq W_{R}, \\
\mathbf{F}_{R} & \text{if } W_{R} \leq 0
\end{cases} \tag{15}
\]

where

\[
\mathbf{F}_{M} = \mathbf{F} (U_{M}), \quad M = L \text{ or } R, \\
\mathbf{F}_{W_{L}} = \mathbf{F}_{L} + W_{L} (U_{W_{L}} - U_{L}), \\
\mathbf{F}_{W_{R}} = \mathbf{F}_{R} + W_{R} (U_{W_{R}} - U_{R}). \tag{16}
\]

The states \(U_{W_{L}}\) and \(U_{W_{R}}\) are given by

\[
U_{W_{M}} = h_{M} \left( \frac{W_{M} - h_{M}}{W_{M} - W_{*}} \right)^{\frac{1}{\psi_{M}}} \tag{17}
\]

For nonzero values of \(h_{L}\) and \(h_{R}\) (i.e., for nonzero water depth) the wave speed estimates are suggested by Toro (2001) as

\[
W_{L} = u_{L} - c_{q} q_{L}, \tag{18}
\]

\[
W_{R} = u_{R} + c_{q} q_{R},
\]

\[
W_{*} = \frac{1}{2} (u_{L} + u_{R}) + c_{L} - c_{R}, \tag{18}
\]

where

\[
q_{M} = \begin{cases} 
1 & \text{if } h_{*} < h_{M}, \\
\sqrt{2} \left[ \frac{h_{*}}{h_{M}} \right] & \text{if } h_{*} \geq h_{M}
\end{cases} \quad M = L \text{ or } R, \tag{19}
\]

\[
h_{*} = \frac{1}{8} \left[ \frac{1}{2} (c_{L} + c_{R}) + \frac{1}{4} (u_{L} - u_{R}) \right]^{2}, \tag{20}
\]

and the gravity wave speed

\[
c_{M} = \sqrt{gh_{M}}, \quad M = L \text{ or } R. \tag{21}
\]

e. Numerical stability

The stability in the finite-volume formulation is restricted by the Courant–Friedrichs–Lewy (CFL) condition

\[
\Delta t \leq \frac{\min(r_{i,j})}{2.2 \max[(c + \sqrt{u_{i}^{2} + v_{j}^{2}}),]}, \tag{22}
\]

where \(r_{i,j}\) represents the whole set of distances between every center point of cell \((i, j)\) and those of its four adjacent cells.
f. Numerical experiments and convergence tests

Before proceeding further, we illustrate by numerical examples the convergence characteristics of the proposed SFV scheme for smooth problems. The two sample problems consist of a simple scalar advection equation, the simplest possible hyperbolic system, and the linearized shallow water equations. The first confirms the basic soundness of the SFV machinery, while the second tests the interfacial conditions. Both tests are amenable to analytic solution, which we use to compute errors. As expected, the errors and the convergence rate depend on the number of elements and on the order of interpolation. To compare the SFV solution of the full shallow water equations with a well-established finite-difference model solution, we simulate the westward propagation of a monopole vortex in an idealized basin.

1) Rotation of a Gaussian hill

The scalar problem we consider is the standard advection of a passive tracer in a steady velocity field that corresponds to solid-body rotation. The equations and initial conditions are

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0, \quad h(x, y, 0) = e^{r^2/2},$$

where $r^2 = \left( x - \frac{1}{4} \right)^2 + \left( y - \frac{1}{2} \right)^2$.

Here, $h$ is interpreted as the concentration of a tracer. The $e$-folding length scale is set to $l = 1/16$. Periodic boundary conditions are imposed on all sides of a square basin with unit length, and $\omega$ is set to $2\pi$ so that the period of rotation is 1. After one revolution the exact solution coincides with the initial Gaussian distribution.

Figure 3 shows the error and convergence rates of the SFV solution after a single rotation of the Gaussian hill. The interpolation order is held fixed for each of the curves appearing in Fig. 3a, and the number of elements is increased; this corresponds to the so-called $h$ refinement. The constant slope for fixed $N$ shows that the errors decrease algebraically fast as the number of elements increases; the slope of the line is approximately $N^{-1}$. Higher-order polynomial interpolation results in smaller errors than low-order polynomials for the same grid size (cell size, $\Delta x$) regardless of grid resolution. Figure 3b shows the exponential decrease in the error with the order of the interpolation polynomial for fixed element partition $K$ whose square is the total number of elements.

2) Standing wave in a square basin

To illustrate the accuracy and convergence rate of the SFV method on a hyperbolic system, we apply it to solve the linearized inviscid shallow water equations.

$$u = -\omega \left( y - \frac{1}{2} \right), \quad v = \omega \left( x - \frac{1}{2} \right).$$

Here, $u$ and $v$ are the horizontal velocities in the $x$ and $y$ directions, respectively, and $\omega$ is the angular frequency.
The problem consists of a standing wave sloshing in a rectangular basin of unit width. The boundary conditions are impermeable walls, and the initial conditions are \( u = v = 0 \), and \( h = H + \cos(\pi x) \cos(\pi y) \), where \( H \) is the resting layer thickness. The linearized equations and the solution to the problem as posed are

\[
\begin{align*}
\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + g \frac{\partial h}{\partial y} &= 0, \\
\frac{\partial h}{\partial t} + H \frac{\partial u}{\partial x} + H \frac{\partial v}{\partial y} &= 0, \\
\end{align*}
\]

with \( u = \frac{g}{2H} \sin(\pi x) \cos(\pi y) \sin(\pi \sqrt{2gHt}) \) and \( v = \frac{g}{2H} \cos(\pi x) \sin(\pi y) \sin(\pi \sqrt{2gHt}) \) (24).

We have normalized the gravity coefficient to 1 and set \( H \) to 2 so that the wave period is unity. We have used very small time steps in our calculations so that the error is dominated by the spatial discretization errors even at the highest spatial resolution. The numerical and analytical solutions were compared at \( t = 1 \) (1 wave period), and the root-mean-square error (\( L_2 \) norm) in surface height is computed. Figure 4 shows the error and convergence rate of the SFV solution for various orders of the interpolating polynomials. Again the curves in Fig. 4a, for fixed \( N \), are straight lines indicative of high-order algebraic convergence rates for \( h \) refinement; the slope is approximately given by \( N + 1 \). Figure 4b shows spectral convergence as the order of the interpolation polynomial increases for fixed element partition \( K \).

The previous experiment was performed on structured elements. The convergence rates can deteriorate when the elements deviate from uniform squares. We repeated the experiment on unstructured elements (Fig. 5). Errors from the unstructured elements are larger than those from the structured ones, although the curves are still straight lines (Fig. 6).

3) WESTWARD PROPAGATION OF MONOPOLE VORTEX

The SFV solution of the full shallow water equations has also been compared with a finite-difference solution. Milliff and McWilliams (1994) studied the reflection of a monopole vortex off the western coast of an idealized ocean basin in order to explain the role that boundary pressure plays in coupling coastal and interior dynamics. We use the same boundary conditions and physical parameters as Milliff and McWilliams. The basin size is \( 3600 \times 2800 \) km\(^2\), the basin depth is 1000 m, and the reduced gravity is 0.081 m\(^2\) s\(^{-1}\). The Coriolis parameter is \( f = f_o + \beta y \), where \( f_o = 9 \times 10^{-5} \) s\(^{-1}\) and \( \beta = 1.8 \times 10^{-11} \) m\(^{-1}\) s\(^{-1}\) at the central latitude.

We start the motion with an initial monopole vortex in gradient-wind balance (Milliff and McWilliams 1994). Figure 7 shows the initial condition of dynamic pressure \( (\rho = g' h) \). The initial monopole vortex profile is Gaussian, and the initial velocity field is in steady, \( f \)-plane gradient-wind balance with the initial dynamic pressure field. After the initial adjustment, the monopole vortex propagates mostly westward in the form of Rossby waves. The reflection of the vortex off the western boundary triggers Kelvin waves that propagate along the sides of the basin. Figure 8 compares the finite difference and the SFV solutions at day 80. The two models give the same phenomenology.

3. Flux-corrected transport algorithm

High-order schemes generally produce Gibbs oscillations in regions of unresolved gradients in the solution (Boyd 2001; Durran 1999; Levin et al. 1997). These oscillations may amplify via nonlinear mechanisms and may lead to a breakdown of the simulation. A further complication is the production of unrealistic values, such as negative density or chemical concentration, outside the range anticipated by the continuum equations. In our particular applications where the ocean is represented by layers of varying density, a concern is the preservation of the positivity, or more precisely the non-negativity, of the layer thickness. The aforementioned problems are best addressed by preventing the generation of spurious ripples so that no spurious extrema are produced; this is commonly achieved by applying flux limiting or flux correction. We have adopted the FCT approach, initially proposed by Boris and Book (1973) and generalized to multidimensions by Zalesak (1979), because of its geometric and algorithmic flexibility. In this section we summarize the FCT algorithm as modified to accommodate the SFV scheme for the shallow water equations and discuss issues related to limiting a system of equations. We finish this section by presenting an SFV simulation involving a discontinuity.

a. FCT algorithm for the SFV

The FCT algorithm requires the calculations of two sets of fluxes: low-order fluxes that yield a monotonic
Fig. 4. Standing wave problem of the linear shallow water equation. We plot $L_2$ norm of the global error of layer thickness for (a) fixed order $N$ and (b) fixed element partition $K$. $\Delta x$ is the average grid size, and $N \times N$ points are used for the interpolation of variables within an element; $p$ is the order of convergence for grid size refinement, $\text{error} \propto (\Delta x)^p$.

but diffused solution and high-order fluxes to correct for the low-order truncation errors where the solution is smooth and oscillation free. The high-order fluxes are computed by the SFV method. The low-order flux calculation is based on the Godunov scheme (Godunov 1969; LeVeque 1992), which regards variables as averages within cells, and where fluxes are computed by solution of the Riemann problems. The Godunov scheme is stable and locally conservative; it guarantees the monotonicity of the solution at the end of each time step and propagates discontinuities with the right velocities and without any spurious oscillations. The main steps of the spectral finite-volume FCT algorithm are as follows:

1) Evaluate the field of cell averages ($\overline{U}$) corresponding to the initial condition.

2) Compute monotonic upwind fluxes corresponding to the low-order scheme. The low-order fluxes are denoted by $E_L$ and $G_L$ in the $x$ and $y$ directions, respectively. Calculate the advection terms defined in Eq. (8), ADV ($E_L$, $G_L$).

3) Advance cell averages in time using the low-order

Fig. 5. Comparison of (a) structured and (b) unstructured elements.

Fig. 6. The $L_2$ norm of the global error in layer thickness for fixed order $N$ on structured (solid line) and unstructured (dashed line) elements.
fluxes to obtain the low-order transportive and diffusive solution $\mathbf{U}^{nd}$:

$$\mathbf{U}^{nd} = \mathbf{U}^{n} + \Delta t \text{ADV}(\mathbf{E}_L, \mathbf{G}_L). \quad (25)$$

4) Compute the transportive high-order fluxes $\mathbf{E}_H$ and $\mathbf{G}_H$ corresponding to the spectral finite-volume method. Compute the antidiffusive fluxes $\mathbf{E}_A = \mathbf{E}_H - \mathbf{E}_L$ and $\mathbf{G}_A = \mathbf{G}_H - \mathbf{G}_L$.

5) Limit $\mathbf{E}_A$ and $\mathbf{G}_A$ based on the transportive-diffusive field $\mathbf{U}^{nd}$:

$$\mathbf{E}_{AC} = C_e \mathbf{E}_A, \quad \mathbf{G}_{AC} = C_g \mathbf{G}_A,$$

$$0 \leq C_e \leq 1, \quad 0 \leq C_g \leq 1. \quad (26)$$

The determination of the limiting factors $C_e$ and $C_g$ is described in detail by Zalesak (1979).

6) Compute the source terms $\mathbf{S}$, such as Coriolis force, wind stress, and friction.

7) Advance cell averages based on the limited antidiffusive fluxes and the source terms in time:

$$\mathbf{U}^{n+1} = \mathbf{U}^{nd} + \Delta t \text{ADV} (\mathbf{E}_{AC}, \mathbf{G}_{AC}) + \Delta t \mathbf{S}. \quad (27)$$

8) Reconstruct point values from the cell averages at the new time level $\mathbf{U}^{n+1} = A^{-1} \mathbf{U}^{n+1}$, where $A$ is the cell-averaging matrix for each spectral element.

9) If the target time is not achieved go to step 2.

The time-marching scheme in steps 3 and 7 is the Euler forward scheme. Because the RK3 scheme is a combination of three Euler schemes with positive weighting factors, the time-marching scheme can be extended to the RK3 scheme by repeating steps 1 to 7 in each of three Euler schemes. The RK3 scheme with the FCT algorithm is used throughout the numerical experiments.

b. Limiting for a system of equations

The FCT algorithm gives excellent results for a scalar equation such as the two-dimensional passive-tracer advection Eq. (23). However, when we try to apply the algorithm to a system of equations, there is no obvious limiting procedure. There are various extensions of the FCT algorithm for such systems, with each extension offering its own set of problems and disadvantages. Three of these are

1) Independent FCT (Tóth and Odstrčil 1996) treats each equation independently; the limiting factors ($C_e$ and $C_g$) are calculated and applied for each of the three components of the flux vectors ($\mathbf{E}$ and $\mathbf{G}$) individually. This approach is less diffusive for each variable than the other alternatives but produces
phase errors. The errors lead to an excessive amount of rippling in the presence of strong discontinuities.

2) Synchronous FCT (Löhner et al. 1987; Schär and Smolarkiewicz 1996) uses the same limiting factor for all equations; the limiting factors are calculated individually for each component, and the minimum is applied to all components. This procedure produces fewer phase errors but leads to a diffused solution.

3) Hybrid FCT (Bleck and Boudra 1986) uses the FCT algorithm to solve the continuity equation and activates viscosity terms in the momentum equations. This approach has been used in isopycnal layered ocean models.

c. Supercritical channel flow

We explore the FCT portion of our model by simulating the deflection of a supercritical flow by a channel constriction. A flow is called supercritical when the advection speed is greater then the gravity wave speed so that the Froude number, $Fr = |u|/\sqrt{gh}$, is greater than 1. When a supercritical flow encounters a sudden change in channel cross section, through a boundary wall constriction on one side wall of a channel as depicted in Fig. 9, an oblique hydraulic jump (discontinuity) originates at the constriction.

The equations solved are the fully nonlinear shallow water equations without source terms. The initial conditions over the entire domain including the inlet at $x = 0$ are $h_0 = 1.0 \text{ m}$, $u_0 = 8.57 \text{ m s}^{-1}$, and $v_0 = 0 \text{ m s}^{-1}$; this corresponds to $Fr = 2.74$ at the inflow boundary. The rest of the boundary conditions are supercritical outflow at $x = 40 \text{ m}$ and no-normal flow along the side walls. No boundary conditions are required on the outflow boundary since the flow is supercritical. For a constriction angle of 8.95°, the analytical solution is $h_d = 1.5 \text{ m}$, $|\mathbf{u}| = \sqrt{u^2 + v^2} = 7.9556 \text{ m s}^{-1}$, and $Fr_d = 2.075$ downstream of the jump; the angle between the original flow direction and the jump is 30°.

A straightforward application of the SFV method without FCT in this test problem is not possible because the high-order method produces enormous oscillations near the discontinuity. Figure 10 shows the water depth contours after the shock is fully developed; the grid divides quadrilateral elements that contain $6 \times 6$ cells ($16 \times 12$ elements, $N = 6$, and $96 \times 72$ grid cells). Three solutions are shown corresponding to the three FCT approaches discussed earlier. In all cases, the shock is well resolved within the span of a few cells and its angle is 30°. The independent application of FCT produces small amplitude ripples near the shock, whereas the synchronous and hybrid application of FCT do not. The errors in water height over the entire domain by the synchronous application ($2.08 \times 10^{-2}$ m) is slightly smaller than those of the independent application ($3.21 \times 10^{-2}$ m) and hybrid application ($2.34 \times 10^{-2}$ m) because the synchronization of phase errors among the flux components reduces oscillation amplitudes downstream of the shock. Experimentation with the approximate Riemann solver has shown that HLLC captures the strong shock better than the Roe solver (Giannakouros et al. 1994), at least for this problem, with the latter producing large oscillations at element boundaries.

We have performed a convergence test for the synchronous FCT simulation by increasing the interpolation order ($p$ refinement; shown in Fig. 11) and by increasing the number of elements ($h$ refinement; shown in Table 1). One can anticipate that the $h$-refinement strategy is more advantageous here because of the discontinuity. As the number of grid cells increases, the root-mean-square (rms) error in water depth decreases, and the spatial average of water depth $\bar{h}$ and water speed $|\mathbf{u}|$ for the downstream of the shock, that is, triangular area, approach the analytic solutions (Table 1). Figure 11 shows that the error decreases as the average grid size decreases. The numerical scheme is converging at the same rate regardless of the interpolation order; the slopes of the lines are about 1.5 and the errors are almost equal for the same grid size. Note that the scheme is first order whenever the FCT algorithm activates its limiters. Since the rms error includes the region of the discontinuity in its estimate, the convergence rate is necessarily of first order.

4. Reduced-gravity ocean model

We apply the SFV shallow water solve configured in reduced-gravity mode to simulate the upper-ocean response to wind forcing. The ocean is thus assumed to consist of a single active layer of fluid of constant density and variable layer thickness $h(x, y, t)$, overlying a denser, deep, and motionless layer: the motion of the upper layer represents the first baroclinic mode. All ther-
modynamic effects are neglected. The interface between the two fluid layers is a material surface that represents the permanent thermocline. The FCT algorithm is applied only to the continuity equation to keep the layer thickness positive. A Laplace-type horizontal diffusion term in Eq. (4) is employed on the momentum equation. Two configurations are considered: the canonical dou-

**Table 1.** Convergence of numerical solution by the spectral finite-volume method with synchronous FCT for the supercritical channel flow with $N = 6$. The $L_2$ error in $h$ is the root-mean-square of the differences between predicted water depth and analytical solution over entire domain; $\bar{h}_d$ is the spatial average predicted value of water depth behind the shock; and $|\bar{u}_d|$ is the spatial average predicted value of water speed behind the shock.

| No. of grid cells | $L_2$ error in $h$ | $\bar{h}_d$ | $|\bar{u}_d|$ |
|------------------|-------------------|------------|-------------|
| 24 $\times$ 18   | $1.6871 \times 10^{-1}$ | 1.4858    | 7.9612      |
| 48 $\times$ 36   | $6.0400 \times 10^{-2}$ | 1.4913    | 7.9591      |
| 72 $\times$ 54   | $3.2281 \times 10^{-2}$ | 1.4949    | 7.9561      |
| 96 $\times$ 72   | $2.0819 \times 10^{-2}$ | 1.4962    | 7.9554      |
| 144 $\times$ 108 | $1.1581 \times 10^{-2}$ | 1.4977    | 7.9547      |
| Exact solution   | 1.5 m             | 7.9556 m $s^{-1}$ |
ble-gyre circulation in a square basin and the circulation in the North Atlantic basin. In both cases, we are particularly interested in the regime where the active layer vanishes.

a. Wind-driven circulation in a square basin

Linear analysis of the circulation in a two-layer, square midlatitude ocean basin shows the upper-layer flow pattern to be controlled by a single nondimensional number $\lambda = LW/g'\rho H^2$ (Parsons 1969), where $L$ is the basin width, $W$ the maximum wind stress, $g' = g\Delta\rho/\rho$ the reduced gravity, and $H$ the mean upper-layer thickness.

The simulations are performed in a square basin of width $L = 2040$ km with free-slip walls. The initial depth of the active layer is 150 m and the reduced gravity is set to 0.015 m s$^{-2}$. The Coriolis parameter is $f = f_0 + \beta y$, where $f_0 = 8.37 \times 10^{-5}$ s$^{-1}$, bottom drag ($\alpha$) is $10^{-8}$ s$^{-1}$, and the time step is 1200 s. The circulation is forced with a double-gyre zonal wind stress of the form

$$\tau_z = -\tau_0 \cos(2\pi y/L),$$

where $\tau_0 = 0.03$ N m$^{-2}$. This forcing yields an idealized subtropical gyre in the southern half of a rectangular domain and a subpolar gyre in the northern half, often referred to as a double-gyre system. With this set of parameters, the nondimensional number $\lambda$ is 0.18, larger than the critical value $\lambda_c$. The lower layer vanishes at the center of the subpolar gyre and a strong boundary current is formed along the northern and western boundary boundaries, and along the outcropping line in the subpolar gyre.

A number of numerical experiments were carried out to test the sensitivity of this outcropping test to different order of interpolation, grid size, and level of Laplacian dissipation. We focus here on the order $N = 3$. Figure 12 shows the instantaneous upper-layer thickness at year 6. The average grid size varies from 20 to 5 km and the lateral viscosity from 300 to 50 m$^2$ s$^{-1}$. The Rossby deformation radius is $R = \sqrt{g H f_0} \approx 20$ km in the subtropical gyre. The simulation with a coarse resolution and high viscosity produces a diffusive solution in Fig. 12a like a linear model (e.g., Huang and Flierl 1987). As grid size and viscosity decrease, the width of the boundary currents become thinner and the eastward jet meanders more energetically in Figs. 12b and 12c. The simulation with smallest grid size (5 km) and viscosity (50 m$^2$ s$^{-1}$) is enough to resolve the scales of interest in Fig. 12d. The pattern of upper-layer thickness is unsteady with a meandering eastward jet, and an oscillating subtropical gyre (Chang et al. 2001). Eddies pinch off from the meandering jet in the subpolar gyre and propagate into the outcrop region with their trapped subtropic waters. The layer thickness is always positive and the front along the outcropping line is resolved in all simulations.

b. North Atlantic Ocean

In this section, we demonstrate the model’s ability to handle unstructured grids, complex coastlines, and vanishing layer thickness by simulating the wind-driven circulation in the North Atlantic Ocean. Given the latitudinal extent of the basin, from 20°S to 75°N, the equations are solved in their spherical form. The North Atlantic Ocean and the equatorial Atlantic Ocean are divided into 792 elements (Fig. 13) with each element containing $5 \times 5$ cells. The grid size varies from 6 km in the Gulf Stream region to 208 km in the equatorial Atlantic Ocean, with an average of 47 km.

The initial thickness of the upper layer is 150 m, and the reduced gravity is 0.06 m$^2$ s$^{-1}$. The flow is driven by the long-term mean winds of the National Centers for Environmental Prediction–National Center for Atmospheric Research (NCEP–NCAR) reanalysis. Integration starts from rest, and free-slip boundary conditions are applied. A bottom drag of $10^{-8}$ s$^{-1}$ is used to balance the wind stress forcing. The time step is 720 s. A variable eddy viscosity is employed in order to provide larger friction in regions of high velocity shear and large grid size, while weaker friction is used in the ocean interior and region of small grid size. The horizontal viscosity is parameterized
as \( \nu = \nu_0 + 2.0 \Delta x \Delta y |D| \), where \( \nu_0 \) is background horizontal viscosity, \( \Delta x \Delta y \) is the area of the cell, and \( |D| \) is the absolute value of the total deformation of the velocity field (Smagorinsky 1963; Deardorff 1971):

\[
|D| = \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right]^2 + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^{3/2}.
\]

(Bleck et al. 1992) used the proportionality constant 2.0, and Griffies and Hallberg (2000) chose the constant 1.0 for their simulations. The background horizontal viscosity is set to 300 m\(^2\) s\(^{-1}\).

Figure 14a shows instantaneous contours of upper-layer thickness after 3 yr of simulation from rest. There are both a cyclonic subpolar gyre above 50\(^\circ\)N and an anticyclonic subtropical gyre below 50\(^\circ\)N. The upper layer vanishes at the center of the subpolar gyre, southeast of Greenland (Fig. 14a). The East Greenland Current and the Labrador Current are developed along the northwestern boundaries. Another boundary current is evident along the outcropping line about 55\(^\circ\)N. The latitude of the outcropping line depends on the initial choice of upper-layer thickness. A loop current forms in the Gulf of Mexico and sheds eddies. The Gulf Stream is intensified along the western boundary, and it separates correctly near Cape Hatteras, North Carolina.

To test sensitivity to grid size, we repeat the simulation on a fine grid. Each element in Fig. 13 is divided into four elements to obtain the fine-resolution grid. The average grid size becomes 23.5 km and all other parameters are the same. In this fine resolution, the western boundary currents become thinner and the Gulf Stream meanders after it separates from the coast (Fig. 14b).

The locations of the boundary currents, the outcropping line, and the gross patterns of circulation are the same as in the coarse-resolution counterpart.

5. Conclusions and discussion

We have developed a model based on the spectral finite-volume method to solve the shallow water equations. The geometric flexibility and high accuracy of this method are built upon the two-tier grid of globally unstructured elements and locally structured cells. Numerical tests on problems with smooth solution confirm the high-order convergence rate of the method. Although the method can be extended to arbitrarily high order in \( N \), practical computational and accuracy considerations for our applications favor a limit of approximately \( N = 7 \).
Gibbs oscillations in underresolved simulations, and where the solution contains discontinuities, are effectively eliminated with the FCT algorithm, as demonstrated in the supercritical channel flow and the reduced-gravity simulations. Experimentation with the application of FCT to a system of equations indicates that the synchronous application of the correction factors to all fluxes improves robustness at the expense of increased numerical dissipation. The compromise of limiting FCT to the continuity equation, while counting on viscous dissipation to damp the momentum equations, seems to work well in our reduced-gravity ocean modeling experiments. The model is able to reproduce steep outcropping, and the circulation exhibits the major ocean currents and eddies expected.

The computation of second derivatives are not trivial in a high-order finite-volume method, which complicates the implementation of viscous operators. Instead of using the finite-volume approach we used the spectral element method to calculate the horizontal viscosity terms. All variables are interpolated on the Gauss–Lobatto–Legendre points and the second derivatives are calculated in the weak form (Iskandarani et al. 1995). The discontinuous Galerkin method was also tried to calculate the horizontal viscous fluxes and it did not improve our simulations.

We now that the SFV method in our experience is more expensive than methods based on variational methods like the spectral element method or the discontinuous Galerkin methods. The reason is the increase in computational cost due to the flux calculations at cell edges; these require expensive high-order interpolation to nonstandard collocation points. Operation count is proportional to $N^4$ in the SFV method and $N^3$ in variational methods. However, the increased computational cost is small when considering the relatively low interpolation order. Furthermore, the ability to use standard TVD schemes developed for other finite-volume methods more than compensates for the extra cost. One additional mitigating factor is that most calculations are entirely local to an element and can proceed independently. This makes the method well suited for parallel computers. In fact, a parallel version of the model was developed using techniques borrowed from Curchitser et al. (1998).

The performance of the FCT scheme can be improved by, for example, including a discriminator to distinguish between smooth extrema and true discontinuities (Schechetkin and McWilliams 1998). Another model improvement is to change the interpolation order dynamically according to the evolving solution. Areas where the limiters are acting frequently would have a lower-order interpolation than where they act more intermittently. These experiments will be pursued in the near future. Our next major goal is to develop a multilayer version of the present model for application to regional and large-scale ocean circulation modeling.

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