Brief review for Exam 1

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Topics/Outline:

1. Introduction to the basics (homework 1)
2. Equation of state (homework 2)
3. Work, energy and stability (homework 3)
4. Heat budget and continuity (homework 4)

1. Introduction to the basics (homework 1)

In homework 1 we reviewed the fundamentals of differential calculus for physical oceanography. A number of processes in the ocean can be described as functions of several variables, which include space (x,y,z) and time t, therefore some of the problems asked you to practice partial derivations.

We defined the (x,y,z) directions as zonal, meridional and vertical, respectively. We also said that the velocity components in this Cartesian coordinate system are 
\[ u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt} \]
and that the accelerations are the derivatives of each velocity component: 
\[ a_x = \frac{du}{dt}, \quad a_y = \frac{dv}{dt}, \quad a_z = \frac{dw}{dt} \]
We calculated vertical temperature gradients in the ocean by using a finite difference scheme, which we used later to find the stability of the water column.

2. Equation of state

The density of the ocean is a nonlinear function of some variables such as pressure, temperature and salinity. One of the most accurate ways to calculate it is via the UNESCO equation, which is a complicated polynomial. This formula is also available if graph format for quick calculations.

Sometimes we can consider that the density of the ocean is a linear function of salinity, temperature and pressure:
\[ \rho = \rho_0 - a(T - T_0) + b(S - S_0) + kP \]
Where \( a \) is the coefficient of thermal expansion, \( b \) is the saline contraction and \( k \) is the pressure compression. Warmer water tends to be lighter while salinity and pressure tend to make the water denser.

3. Work, energy and stability

In homework 3 we considered an ocean of three layers of different densities. We took a water parcel and moved it vertically to examine the changes in work, potential and kinetic energy. When a particle is surrounded by fluid, there are two important forces acting on it:
**Buoyancy**: is the weight of the fluid displaced by the parcel (here the density is the density of the fluid):

\[ B = \rho \nabla g \]

**Weight**: This is simply the product of mass and gravity, but we write the mass of the parcel as the product of its density and volume:

\[ W = \rho_p \nabla g \]

Then, the net force acting on the parcel can be written as:

\[ W - B = (\rho_p - \rho) \nabla g \]

It follows that the direction of the net force depends on the sign of the density difference. If the net force is different than zero, and if the particle moves in the vertical, then it does work. Consider that it has moved a vertical distance \( H \), then the work done would be:

\[ \text{Work} = (\rho_p - \rho) \nabla g H \]

We used this definition of work and the definitions of kinetic and potential energy to find both the velocities and the excursions of the particle as it moves in the water column.

4. **Heat fluxes and continuity**

Before our discussion on the heat budget of the ocean, we reviewed two concepts in recitation: heat and heat flux.

In a nutshell, heat is a form of energy that depends on the temperature of a substance. We defined this heat (in units energy) as:

\[ Q = mc_p T \]

On the other hand, the heat flux is the amount of heat transferred from one place to another, per unit area and time. So we wrote:

\[
\text{Heat flux} = \frac{\text{heat}}{\text{area} \cdot \text{time}} = \frac{mc_p T}{At} = \frac{\rho(\Delta H)c_p T}{At} = \rho c_p T \frac{H}{t} \\
= \text{(density)(specific heat)(temperature)(velocity scale)}
\]

Which we saw in the bulk aerodynamic formulations from Stewart’s textbook on latent and sensible heat fluxes:

**Sensible heat flux**:

\[ Q_s = \rho c_p C_h U_{10}(T_s - T_a) \]

This flux is from ocean to air if \( T_s > T_a \) and \( U_{10} > 0 \)
Or from air to ocean if $T_s < T_a$ and $U_{10} > 0$

*Latent heat flux*

$$Q_L = \rho L U_{10} (q_s - q_a)$$

This is a one-way only flux: from ocean surface to atmosphere because $q_{sat} > q_a$. It is also possible to write this as the *evaporative flux*, which can be written as $\rho E L$ where $E$ is the evaporation rate in m/s.

*Longwave radiation:*

$$Q_{LW} = \sigma T^4$$

*Shortwave radiation:*

Comes from the sun. It can also be calculated via Stephan-Boltzmann formula but they will usually just give it to you as the solar constant, which is around 342 W/m$^2$.

*Continuity:*

This is the solution for the problem on conservation of mass (problem 4, part II).

In a mid-ocean region in the northern hemisphere the east-west current is zero, and the north-south current is given by:

$$v = \frac{yz}{LT}$$

Where $L$ is a length scale and $T$ is a time scale (both are constants).

a) In a y-z plot, draw arrows at three different depths and three different values of $y$ to indicate how the speed changes with depth and in the north-south direction.

In this problem they are asking to draw vectors of the north-south velocity $v$, on a y-z plane. **Always make sure that you denote z as negative** in physical oceanography problems unless the problem indicates otherwise. Choosing z as positive will completely change your solution.
Here is one (fast) way to do it:

1. **Draw your y-z plane so that you know what’s positive and what’s negative:**

   ![Diagram of y-z plane]

   In this case, z is always negative and y is always positive.

2. **Find out the direction of the velocity (sign):**

   The velocity is given by
   \[ v = \frac{yz}{LT} \]

   Where \( L \) and \( T \) are positive constants. The sign of the velocity is:
   \[ v = \frac{yz}{LT} \rightarrow (+)(-)(+)(+) = (-) \]

   So, we get that \( v \) is negative for the section of the y-z plane we chose. This implies that *all* the arrows will point towards the south direction (or, *in our plot*, towards the left).

3. **Check the magnitude:**

   We just found that the arrows will go towards the left. Now let’s examine what controls the *length* of these arrows. Look at the equation again without considering the signs:
   \[ v = \frac{yz}{LT} \]

   We can see that the *magnitude* of \( v \) is directly proportional to the magnitude of \( y \) and \( z \). So the arrow length grows as it gets deeper, and as \( y \) increases. This also tells us that the smallest arrow will be located on the upper left section where the magnitudes of \( y \) and \( z \) are small. And the largest arrow will be on the lowest right corner where the magnitude of \( y \) and \( z \) are big with respect to the other points. The velocity field looks like this then (not to actual scale):
(b) Based on your sketch in (a), is the vertical velocity at a specific depth upward or downward? Why?

In order to answer this question we can solve for $w$ using the continuity equation and find its sign the same way we did for the first part. But it is also possible to tell this from intuition: the water at depth is flowing faster than the water near the surface, so it is reasonable to say that some water from the upper layers is feeding that strong flow below. This would imply that the vertical velocity is downwards.

(c) Using the continuity equation with density assumed to be constant, determine the magnitude of the vertical velocity

We do the math to integrate the continuity equation:

$$\frac{dv}{dy} + \frac{dw}{dz} = 0$$

$$\frac{d}{dy} \left( \frac{yz}{LT} \right) + \frac{dw}{dz} = 0$$

$$\frac{z}{LT} + \frac{dw}{dz} = 0$$

$$\int dw = \int -\frac{z}{LT} dz$$

$$w = -\frac{z^2}{2LT}$$

For negative $z$, $w$ is negative, so we confirm that the velocity is downwards. Also, note that this vertical velocity is only a function of depth. We can use this expression to find the vertical velocity at any depth.