1. Review of momentum equation and forces (pressure gradient, Coriolis, and gravity):

\[
\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \bullet \nabla) \vec{V} = \vec{V} \times (2\Omega_o + \Omega_r) - \frac{1}{\rho} \nabla p - g \vec{k} \tag{1}
\]

\[
\frac{du}{dt} = f v - \bar{f} w + \frac{uv}{r} \tan \phi - \frac{uv}{r} - \frac{1}{\rho} \frac{\partial p}{\partial \phi} \tag{1a}
\]

\[
\frac{dv}{dt} = -fu - \frac{u^2}{r} \tan \phi - \frac{uv}{r} - \frac{1}{\rho} \frac{\partial p}{\partial \phi} \tag{1b}
\]

\[
\frac{dw}{dt} = f u + \frac{u^2 + v^2}{r} - \frac{1}{\rho} \frac{\partial p}{\partial \phi} - g \approx -\frac{1}{\rho} \frac{\partial p}{\partial \phi} - g \tag{1c,d}
\]

Where, \( \vec{i}, \vec{j}, \vec{k} \), unit vectors in x, y, z directions; \textbf{position vector} from the center of the Earth to point \((x,y,z)\), radial vector from the axis of the Earth to point \((x,y,z)\); \textbf{3D-speed} \( \vec{V}(x(t), y(t), z(t), t) = u \vec{i} + v \vec{j} + w \vec{k} \) \((m s^{-1})\); \textbf{earth rotating angular speed} \( \Omega_o = \Omega \cos \phi \vec{j} + \Omega \sin \phi \vec{k} \); \textbf{Coriolis parameter} \( f = 2\Omega \sin(\phi) \), \( \bar{f} = 2\Omega \cos(\phi) \); \textbf{operator} \( \nabla = \vec{i} \partial / \partial x + \vec{j} \partial / \partial y + \vec{k} \partial / \partial z \); angular speed of the axis of the Earth \( \Omega = 2\pi / T_{str} = 7.292 \times 10^{-5} \text{s}^{-1} \); length of 1 sidereal day \( T_{str} = 24 \times 3600 \times 365.25 / 366.25 = 86164 \text{s} \), \( \phi \) is latitude angle.

Schematic diagram for angular speed versus speed, altitude, and latitude.
The acceleration \( \frac{d\vec{V}}{dt} \) of a float in Lagrangian system (follow the track, \( x, y, \) and \( z \) are independent on each other, but dependent on \( t \)) can be mathematically written into Eulerian formation \( \left( \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right) \), the field pattern, \( x, y, z, \) and \( t \) are independent on each other) through “chain rule” of partial derivative. The acceleration equals the sum of local acceleration \( \left( \frac{\partial \vec{V}}{\partial t} \right) \) and field acceleration \( \left( (\vec{V} \cdot \nabla) \vec{V} \right) \). For static field, \( \frac{\partial}{\partial t} = 0 \), for spatially even field, \( \vec{V} \cdot \nabla = 0 \). For an object, \( (\vec{V} \cdot \nabla) \vec{V} \), \( \text{and} -\rho^{-1} \nabla p \) vanish; for a float, \( (\vec{V} \cdot \nabla) \vec{V} \) vanishes.

**Forces \([N]\):** The **apparent forces** include Coriolis force, curvature force, and centrifugal force (merged into gravity) introduced into Newton equation when the reference coordinate has acceleration (rotates and/or accelerates in a direction) \(^\wedge\); the **real forces** include pressure gradient force, gravity, and other forces \( \vec{F}_o \) such as frictional force and gravitations of heaven bodies.

**Coriolis force:** \( \vec{F}_c = 2\vec{V} \times \Omega_o \) (2)

Coriolis force is an Earth-rotation induced inertial force applied on a moving object, points to right (left) of motion direction in northern (southern) hemisphere by 90°, and is proportional to the size of speed. Terms that have \( \vec{f} \) are omitted (for mid-high latitudes), which does not affect kinetic energy.

\[
(2\vec{V} \times \Omega_o) \cdot \vec{V} \equiv 0, \text{ does not change kinetic energy.}
\]

**Curvature force:** \( \vec{F}_c = \vec{V} \times \Omega \) (3)

Curvature force, mostly omitted, is an inertial force applied on a moving object on spherical Earth in a local flat coordinate (it vanishes in spherical coordinate), points to right/left of the speed in northern/southern hemisphere by 90°, is proportional to size of speed and latitude’s tangent. \((\vec{V} \times \Omega) \cdot \vec{V} \equiv 0\), changing no kinetic energy. Where **curvature angular speed** \( \Omega_r = r^{-1}(-v \hat{i} + u \hat{j} + utg \phi \hat{k}) \).

**Centrifugal force:** \( \vec{F}_r = \Omega_o \times r \times \Omega_o = \Omega_o \times R_c \times \Omega_o = \Omega_o^2 R_c \) (3,4,5)

**Gravity:** \(-k \frac{GM_e}{r^2} + \Omega_o^2 R_c \approx -g \hat{k} \) (6)

Gravity, the functions of position, is the sum of the true gravitational attraction toward Earth’s center and centrifugal force outward from Earth’s axix. \( G = 6.67 \times 10^{-11} \text{N m}^2/\text{kg}^2 \), the mass of Earth \( M_e \approx 5.97 \times 10^{24} \text{kg} \), \( g_o \approx 9.8 \text{m/s}^2 \), for ocean or atmosphere whose thickness <<\( r, r \approx R_e \approx 6371000 \text{m} \) (the radium of the Earth). \( g \approx g_o + 2.3 \times 10^{-6} z \approx g_o \).

\* The total angular speed of a reference coordinate is \( \omega = \Omega_o + \Omega_r \) (if the origin of the coordinate is at a local point). Where \( \Omega_o = \Omega \cos \phi \hat{j} + \Omega \sin \phi \hat{k} \), \( \Omega_r = r^{-1}(-v \hat{i} + u \hat{j} + utg \phi \hat{k}) \) (vanishes if without local effect). Mark \( d_o \) / \( dt \) as the full derivative in an inertial coordinate while \( d / dt \) in the reference coordinate. \( d_o \hat{r} / dt = d \hat{r} / dt + \omega \times \hat{r} \)

\[
\begin{align*}
   w \hat{k} + \omega \times \hat{r} = \hat{V} + r \Omega \cos \phi \hat{i}, \quad d_o \hat{r} / dt = d \hat{r} / dt + d_o (r \Omega \cos \phi \hat{i}) / dt = d \hat{V} / dt + \omega \times \hat{V} + d (r \Omega \cos \phi \hat{i}) / dt + \omega \times (r \Omega \cos \phi \hat{i}) = \hat{F},
   d \hat{V} / dt = \hat{F} - \omega \times \hat{V} - j \left( \omega \cos \phi - \nu \Omega \sin \phi \right) - \omega \times (r \Omega \cos \phi \hat{i}) = \hat{F} + \hat{V} \times (2 \Omega_o + \Omega_r).
\end{align*}
\]
Geopotential height $Z$: \[ Z = g_o \frac{1}{z} \int_{z_o}^{z} g(z)dz \approx z \ [m] \ (z_o=0) \] (7)

Geopotential energy for per unit mass at height $z$: \[ \Phi(z) = \int_{z_o}^{z} g(z)dz \approx g(z) \ [m^2/s^2] \] (8)

Pressure ($p$) and horizontal pressure gradient force ($F_p$) at a height $z$:
\[
p(x, y, z, t) = p_o(x, y, t) + \int_{z_o}^{z} g \rho_o(x, y, t)dz + \int_{z}^{z_o} gp(x, y, z, t)dz
\approx p_o(x, y, t) + g \rho_o(x, y, t)(z_s - z_o) + \int_{z}^{z_o} gp(x, y, z, t)dz
\]
(9)

Where, $z \in (z_b, z_s)$, $z_b = \delta(x, y, t)$ the bottom height, $z_s = \sigma(x, y, t)$ the surface height of the fluid, $z_o$ is close enough to the surface height and is a reference height above which the density does not change much with height. $p_o$ the pressure at the top of the fluid $p_o \approx 1013hPa \approx const$ (for ocean) or 0 (for atmosphere).
\[
\vec{F}_p = -\rho^{-1}\nabla h P = -\rho^{-1}(i \frac{\partial p}{\partial x} + j \frac{\partial p}{\partial y})
\]
(10)

If $\rho = \rho_o(x, y, t)$ and $p_o = const$ ,
\[
p(x, y, z, t) = p_o(x, y, t) + g \rho_o(x, y, t)(z_s - z)
\]
(11)
\[
\vec{F}_p = -g(i \ n_x + j \ n_y) + g \ k
\]
(12)

Where, $n_x = \partial \sigma / \partial x$, $n_y = \partial \sigma / \partial y$, the slopes of the fluid surface in x, y directions.

2. Simplification of momentum equation with scale analysis

Scale analysis uses the typical values of variables to compare the magnitudes of various terms in the governing equations to simplify the equations by canceling smaller term(s). The typical values include the mean magnitudes and the fluctuation amplitudes of field variables, the temporal and spatial scales, etc.. The simplified equations must be balanced in magnitudes. Simplification based on scale analysis is feasible for linear processes or for those on small temporal and spatial scales, but not for climatic and nonlinear systems in which both force’s size and acting time-space matter.

2-1, Some of the important numbers:

1), Rossby number $R_o \equiv \frac{U}{fL}$

Rossby number represents: the ratios of inertial force scale ($U^2/L$) and Coriolis force scale ($fU$), of rotation time scale ($1/f$) and advection time scale ($L/U$), of relative vorticity scale ($U/L$) and inertial vorticity scale ($f$), and of relative speed scale ($U$) and inertial speed scale ($fL$).

In natural coordinate, basic order balance among Coriolis, pressure gradient, and centrifugal forces is $O\{fU\} + O\{\partial \Phi / \partial n\} + O\{U^2 / R\} = 0$, or
$O\{U(1 + R_o)\} + O\{f^{-1} \partial \Phi / \partial n\} = 0$
For large scale slow flow, $R_o << 1$, Coriolis force dominates; for small scale fast flow $R_o > 1$, Coriolis force can be omitted.

2), Rossby deformation radius $R_D \equiv f^{-1} \sqrt{gH}$ (for future use)

Rossby deformation radius represents: the distance for shallow-water gravity wave to propagate with speed $\sqrt{gH}$ within the rotation period $f^{-1}$; the e-fold length scale for Kelvin-wave amplitude to decay off-shore; the distance for the Coriolis force to be balanced by gravity in a maintaining sea level slope. For long wave $R_d / \lambda << 1$, for short wave $R_d / \lambda >> 1$ ($\lambda$, wavelength).

3), Brunt number (frequency) $N^2 \equiv -g \gamma \ln \rho / \rho \gamma$ (for ocean, from before)

Brunt number and frequency are used for valuation of the stratification stability.

4), Reynolds number $Re \equiv \rho UL / \mu$ (for future use)

$\mu$, dynamic viscosity ($kg/m/S$); $\nu = \mu / \rho$, the kinematic viscosity ($m^2/s$).

Reynolds number gives a measure of the ratio of inertial forces ($\rho U^2 L^2$) to viscous forces ($\mu UL$) and consequently quantifies the relative importance of these two types of forces for given flow conditions. Laminar flow occurs when $Re < 2300$ and turbulent flow occurs when $Re > 4000$. In the interval between 2300 and 4000, laminar and turbulent flows are possible ('transition' flows), depending on other factors. For atmospheric and oceanic motions, $Re > 10^{10}$, turbulence and mixing are common.

2-2, Typical scales of atmospheric and oceanic variables (Table 1 and 2):

<table>
<thead>
<tr>
<th>Type of motion</th>
<th>Horizontal scale L (m)</th>
<th>Vertical scale H (m)</th>
<th>Time scale t (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minute turbulent eddies</td>
<td>$10^3$~$10^4$</td>
<td>$10^2$~$10^3$</td>
<td>1</td>
</tr>
<tr>
<td>Small eddies</td>
<td>$10^3$~$10^4$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Dust devils</td>
<td>$10^4$~$10^5$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Gusts</td>
<td>$10^5$~$10^6$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Tornadoes</td>
<td>$10^6$~$10^7$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Cumulus clouds</td>
<td>$10^7$~$10^8$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Front, squall lines</td>
<td>$10^8$~$10^9$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Hurricanes</td>
<td>$10^9$~$10^{10}$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Synoptic cyclones</td>
<td>$10^{10}$~$10^{11}$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Planetary waves</td>
<td>$10^{12}$~$10^{13}$</td>
<td>$10^2$~$10^3$</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable scale</th>
<th>unit</th>
<th>atmosphere</th>
<th>ocean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal spatial scale</td>
<td>L</td>
<td>m</td>
<td>$10^6$~$10^8$</td>
</tr>
<tr>
<td>Vertical spatial scale</td>
<td>H</td>
<td>m</td>
<td>$10^4$</td>
</tr>
<tr>
<td>Horizontal flow speed</td>
<td>U</td>
<td>m/s</td>
<td>10</td>
</tr>
<tr>
<td>Vertical flow speed</td>
<td>W</td>
<td>m/s</td>
<td>$10^2$~$10^4$</td>
</tr>
<tr>
<td>Advection time scale</td>
<td>L/U</td>
<td>s</td>
<td>$10^3$~$10^5$</td>
</tr>
<tr>
<td>Convection time scale</td>
<td>H/W</td>
<td>s</td>
<td>$10^5$~$10^7$</td>
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<tr>
<td>Mean density</td>
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<td>kg/m$^3$</td>
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</tr>
<tr>
<td>Horizontal pressure scale</td>
<td>$\delta p$</td>
<td>Pa</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Vertical pressure scale</td>
<td>$\Delta p$</td>
<td>Pa</td>
<td>$10^5$</td>
</tr>
<tr>
<td>Coriolis parameter</td>
<td>fo</td>
<td>s$^{-1}$</td>
<td>$10^{10}$</td>
</tr>
</tbody>
</table>

* Estimated from continuity equation: $O(\partial v / \partial z) \sim W / H \sim O(\partial u / \partial x) \sim U / L$

** Estimated from hydrostatic equation: $O(\partial p / \partial z) \sim \Delta p / H \sim gp$
2-3, Simple examples for scaling analysis of governing equations:

Horizontal momentum equations:

\[
\begin{align*}
\frac{du}{dt} &= f v - f w + \frac{uv}{r} \tan \phi - \frac{uw}{r} - \frac{1}{\rho} \frac{\partial p}{\partial x} + F_{ix} \\
\frac{dv}{dt} &= - f u - \frac{u^2}{r} \tan \phi - \frac{vw}{r} - \frac{1}{\rho} \frac{\partial p}{\partial y} + F_{iy}
\end{align*}
\]

Vertical momentum equation:

\[
\frac{dw}{dt} = f u + \frac{u^2 + v^2}{r} - \frac{1}{\rho} \frac{\partial p}{\partial z} - g + F_{iz}
\]

By directly canceling smaller terms according to different orders will produce different approximations such as **hydrostatic balance** by keeping the two biggest terms in (c) with 4th order accuracy, **geostrophic balance** by keeping the two biggest terms in (a and b) with 1st order accuracy.

3, Some approximations:

1), Boussinesq approximation:

Boussinesq approximation, the most general approximation for large scale of weather and climate processes, states that density differences are sufficiently small to be neglected, except where they appear in terms multiplied by \( g \), the gravity acceleration. With Boussinesq approximation, sound waves are neglected since sound waves move via density variation, gradient force becomes a linear one,

2), Hydrostatic approximation:

It is 4th order accuracy approximation from vertical motion equation

\[ \frac{\partial p}{\partial z} = -\rho g \]

3), Geostrophic motion \( V_g \)

For ocean with Boussinesq approximation,

\[ V_g = -\frac{1}{f \rho_o} \nabla_h p \times k \]

\[ \approx -\frac{g}{f} \nabla_h z \times k - \frac{1}{f \rho_o} \nabla_h (\int g \rho dz) \times k \] (use Eq.9)

**Surface geostrophic geostrophic in deep layer**

Geostrophic motion is horizontal, stable, and even fluid motion result from balance between horizontal pressure gradient and Coriolis forces in free ocean with large horizontal scale. In Northern/Southern hemisphere, it moves counterclockwise/clockwise for Low pressure (cyclone) system, or clockwise/counterclockwise for High pressure (anti-cyclone).
4) Thermal-wind $\mathbf{V}_T$

“Thermal-wind” [s$^{-1}$] is the vertical shear of the geostrophic wind or current. With Boussinesq approximation and hydrostatic approximation,

$$\partial \mathbf{V}_g / \partial z = \frac{g}{f \rho_o} \nabla h \rho \times \mathbf{k}$$

(16)

$$\{ \partial \mathbf{V}_g / \partial z = -\frac{1}{f \rho_o} [\partial (\nabla h p) / \partial z] \times \mathbf{k} = -\frac{1}{f \rho_o} [\nabla h (-g \rho)] \times \mathbf{k} = \frac{g}{f \rho_o} \nabla h \rho \times \mathbf{k} \}$$

In Northern/Southern hemisphere, higher temperature (lower density) is on the right/left side along thermal-wind vector.

Exercises

Problems:
Calculate the geotropic currents (sizes and directions) above h2 layer (z>h2) and at bottom (z=h3).

$\mathbf{f}=10^{-4}$ s$^{-1}$, density $\rho_1=1020$ kg m$^{-3}$, $\rho_2=1040$ kg m$^{-3}$. Omit surface pressure.

Solutions:
The depth of the surface (Zo), interface (Z1), and bottom (Zb) changes with x starting at B as,

$Zo(x) = \frac{h1}{L} = \frac{x}{20000}$ (m)

$Z1(x) = h2 + x \frac{h3-h2}{L} = -20m + x \frac{-100m-(-20m)}{20000m} = -20m - \frac{x}{250}$ (m)

$Zb(x) = h3 = -100$ (m)

Pressure above $h2 = -20m$ (z >$h2$) is $P1(x) = \rho_1 g (Zo(x) -z) = \rho_1 g (x/2000 - z)$,

Pressure gradient force is $-\rho_1^{-1} \partial P1 / \partial x = g/20000$

Geostrophic current is $v = (-g/20000)/f = 9.8 \text{m/s}^2 / (20000 \times 10^{-4} \text{s}^{-1}) = 4.9 \text{m/s} > 0$ (northwards)
Pressure at bottom is \( P_b(x) = P_0(x) + \rho_1 g (Z_0(x) - Z_1(x)) + \rho_2 g (Z_1(x) - Z_b(x)) \)
\[ = P_a + \rho_1 g \left( \frac{x}{20000} + 20 \frac{m}{250} + \frac{x}{250} \right) + \rho_2 g \left( 80m - \frac{x}{250} \right), \]
Pressure gradient force is \( -\frac{1}{\rho_2} \frac{\partial P_b}{\partial x} = g \left( \frac{-81\rho_1}{20000\rho_2} + \frac{1}{250} \right) \)
Geostrophic current is \( V = -g \left( \frac{-81\rho_1}{20000\rho_2} + \frac{1}{250} \right) / f = -9.8 m s^{-2} \times \left( \frac{-81 \times 1020 kg m^{-3}}{20000 \times 1040 kg m^{-3}} \right) + \frac{1}{250} \times (10^{-4} s^{-1}) = -2.73 m/s < 0 \) (southward).

Problem: Motion of a falling object

You are on a 747 flying towards the north at an altitude of 40,000 ft and at a speed of 600 km/hr. Suddenly, the plane’s baggage door pops open and your luggage falls out of the plane. When you land, where should you look for your luggage? Ignore friction in the atmosphere. Assume for convenience a constant latitude of 43° N.

a) State in words the force(s) you need to consider in this problem, i.e., the forces acting on the luggage as it falls.

b) Set up the equations needed to solve this problem.

c) Find an estimate for the displacement of your baggage.

Solution:
a) Coriolis force and gravity are the two forces to be considered for the motion of the falling luggage.

b) Equations:
\[ \frac{du}{dt} = fv - \tilde{f} w \quad (a) \]
\[ \frac{dv}{dt} = -fu \quad (b) \]
\[ \frac{dw}{dt} = \tilde{f} u - g \quad (c) \]

Where, \( u, v, \) and \( w \) are speed components in \( x, y, \) and \( z \) directions, respectively. \( f = 2\Omega \sin(\phi), \tilde{f} = 2\Omega \cos(\phi), \Omega = 7.2921 \times 10^{-5} s^{-1}, \phi = 43^\circ N. \)

Initial conditions:
\( t=0, x=x_0=0, y=y_0=0, z=z_0=40,000 ft=12192 m, u=u_0=0, v=v_0=600 km/hr=166.67 m/s, w=w_0=0. \)

c) Hint: For general case \( \tilde{f} \) and \( f \) terms are equally important (e.g., at mid latitudes), and if \( \tilde{f} \) term is kept in Eq.a, it must be kept in Eq.c to keep kinetic energy conservation, otherwise, Coriolis force would do work on the luggage unreasonably. Here is simple solution with omitted \( \tilde{f} \) terms:
\[ \frac{du}{dt} = fv \quad (1s) \]
\[ \frac{dv}{dt} = -fu \quad (2s) \]
\[ dw/dt = -g \]  

Solve (3s)  

\[ dz/dt = w = wo - gt \rightarrow z - zo = wo \cdot t - \frac{1}{2} g \cdot t^2 \]  

With initial conditions t=0, zo=12192m and wo=0,  

\[ z - zo = -\frac{1}{2} g \cdot t^2 \]  

At landing time t=T, z=0, landing time is  

\[ T = \sqrt{2zo/g} = \sqrt{2 \times 12192m / (9.8m/s^2)} \approx 49.88s \]  

From (1s)  

\[ v = f^{-1} du/dt \]  

Substitute (5s) to (2s) with f=const  

\[ d^2 u / dt^2 + f^2 u = 0 \]  

Solve (6s)  

\[ u = C_1 \cos ft + C_2 \sin ft \]  

Substitute (7s) to (5s)  

\[ v = -C_1 \sin ft + C_2 \cos ft \]  

Apply initial conditions (t=0, u=uo=0, v=vo=600km/hr=166.67m/s) into (7s) and (8s)  

\[ C_1 = 0, \ C_2 = 166.67m/s, \ then \]  

\[ \frac{dx}{dt} = u = 166.67m/s \cdot \sin ft \]  

\[ \frac{dy}{dt} = v = 166.67m/s \cdot \cos ft \]  

Set t=T=49.88s, \( f = 2 \cdot \Omega \cdot \sin(\phi) = 2 \times 7.2921 \times 10^{-5} \cdot s^{-1} \cdot \sin(43^\circ) = 9.946 \times 10^{-5} \cdot s^{-1} \), the displacements of the luggage in x and y directions are estimated  

\[ \int_{xo}^{x} dx = x - xo = \int_{0}^{T} (166.67m/s \cdot \sin ft) dt = -166.67m/s \cdot f^{-1} \cdot \cos ft \left|_{t=0}^{t=49.88s} \approx 20.6m \right. \]  

\[ \int_{yo}^{y} dy = y - yo = \int_{0}^{T} (166.67m/s \cdot \cos ft) dt = 166.67m/s \cdot f^{-1} \cdot \sin ft \left|_{t=0}^{t=49.88s} \approx 8317.7m \right. \]  

Comprehensive problem for mass conservation and acceleration:
Describe the current distribution at A, B, and C inside the pipe with the conditions shown in the Figure attached. Give equations to get the time for a parcel starting at D reach C. Each of the cross intersections is circle.

Solution:
By mass conservation, \( \pi r_a^2 U_a = \pi r_b^2 U_c \), 
\[
U_c = \left( \frac{r_a}{r_c} \right)^2 U_a = \frac{1}{4} U_a
\]
d\(U_a/dt = \partial U_a/\partial t = 0.01 \text{ m s}^{-2} \), the flow speed before D is
\[
U_a(t)=U_a(t=0)+ \int_0^t (dUa / dt) \, dt = 0.02 + 0.01t \text{ (m/s)}
\]
The flow speed after E is
\[
U_c = \frac{1}{4} U_a = 0.005 + 0.0025t \text{ (m/s)}
\]
With Lde<<Lec, for a point between D and E, its mean local speed is approximated as
\[
U_{bm} = \frac{U_a+U_c}{2} = 0.0125+0.00625t \text{ (m/s)}
\]
and the acceleration of a parcel starting at D to E is
\[
dU_b/dt = \frac{\partial U_{bm}}{\partial t} + U_{bm} \frac{\partial U_{bm}}{\partial x} \approx \frac{\partial U_{bm}}{\partial t} + U_{bm} \frac{U_c-U_a}{L_{de}}
\]
\[
= -0.00625 + (0.0125 + 0.00625t) \times [0.005 + 0.0025t - (0.02 + 0.01t)]/5 \text{ (m s}^{-2})
\]
The speed of the parcel moving from D to E is about
\[
U_b = U_a + \int_0^t dU_b/dt \, dt \approx 0.02+0.0162125t
\]
If time are T1 and T2 for a parcel to move from D to E and from E to C,
\[
\int_0^{T_1} U_b \, dt = L_{de}, \quad \int_{T_1}^{T_1+T_2} U_c \, dt = L_{ec}, \text{ i.e.,}
\]
\[
0.02T_1+0.0081T_1^2 = 5, \quad 0.005(T_1+T_2)+0.00125(T_1+T_2)^2 = 1000
\]
Solve these two equations we get the required time: \( T_1+T_2 \)