Describing the tide at a point along an estuary (i.e., x = constant)

Tidal elevation at a point in space (at lowest order) = \( \eta(t) = a \cos \omega t \)

Tidal velocity at a point in space (at lowest order) = \( u(t) = U \cos (\omega t - \phi) \)

\( a = \) tidal elevation amplitude (= 1/2 tidal range), \( U = \) tidal velocity amplitude
\( \omega = \) tidal frequency = \( \frac{2\pi}{T} \), \( T = \) tidal period
\( T \approx 12.4 \) hours for \( M_2 \) tide, \( \omega = 1.4 \times 10^{-4} \) s\(^{-1} \)
\( \phi = \) phase of tidal velocity relative to tidal elevation. Greater \( \phi \) means greater “delay” in waveform.

In above example, \( \phi = -\frac{\pi}{2} = -90 \) deg, i.e., velocity leads elevation by ~ 6 hours. (Caution -- sign depends on whether flood is positive or negative). This “standing” wave type of relative phase is common in short tidal estuaries. In a “progressive” wave, \( \phi = 0 \), i.e., velocity and elevation are in phase. (If ebb is positive, \( \phi = 180 \) deg.) Wind waves approaching a beach from offshore are progressive.
Phase of tidal velocity relative to tidal elevation in real estuaries:

In real estuaries, (absolute) relative phase is between 0 and 90 deg and varies within individual estuaries.

Example of real velocity - stage curve for a tidal marsh channel:

X = observation site  Tides in very shallow water can be highly distorted.
Describing the tide along an estuary (i.e., x varying in space)

Tidal elevation and velocity in space (+x into estuary from sea) and time (at lowest order):

\[ \eta(x,t) = a(x) \cos(\omega t - kx) \]
\[ u(t) = U(x) \cos(\omega t - kx - \phi) \]

\( k = \text{wave number} = \frac{2\pi}{L_t}, \quad L_t = \text{tidal wave length} \)
\( c = \frac{\omega}{k} = \text{phase speed} = \frac{L_t}{T} \)

\( k \) and \( \phi \) can be functions of \( x \), but are often approximated as constant.

Ex. York River-ish: If \( c = 10 \text{ m/s} \), then \( \eta(x = 0) \) (mouth) and \( \eta(x = 40 \text{ km}) \) (West Point) are:

\[ \begin{align*}
\text{Time} & \quad \omega t/(2\pi) \\
\text{Mouth} & \quad -0.5, 0.5, 1, 1.5, 2 \\
\text{West Point} & \quad -0.5, 0.5, 1
\end{align*} \]

\( kx = \text{phase of tide relative to mouth of estuary} \).

Phase \( kx \) (delay of wave form) increases into the estuary.

Spatial variation in tidal amplitude and phase along real estuaries:

\( kx \) is either dimensionless (i.e., radians) or in degrees. \( \pi \) radians = 180 deg.
Cross-sectionally averaged momentum equation along a tidal estuary or embayment:

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x} - \frac{\tau_b}{\rho h} \]

\( u \) = cross-sectionally averaged velocity

\( \tau_b \) = bottom drag, \( \rho \) = water density

Local acceleration + advective acceleration = pressure gradient + bottom friction

(Note: Pressure gradient is barotropic -- i.e., \( \partial \rho / \partial x \) is unimportant)

Cross-sectionally integrated continuity equation:

\[ \frac{b \partial h}{\partial t} = -\frac{\partial}{\partial x} (whu) \]

Rate of change in cross-sectional area of estuary = along-channel convergence in flux in channel

(outside the channel the along-channel velocity is assumed to be negligible)

Scaling Momentum

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x} - \frac{\tau_b}{\rho h} \]

Local acceleration + advective acceleration = pressure gradient + bottom friction

Bottom stress: \( \tau_b = \rho c_d u \cdot |u| = \rho c_d \{8/(3\pi)\} U \cdot U \)  "Linearization of quadratic velocity formulation"

Drag coefficient: \( c_d = 0.01 \) to 0.003 depending on roughness scale

Friction coefficient: \( r = c_d \{8/(3\pi)\} U \cdot h \)

Relative importance of terms: \( O(\cdot) \) indicates "order of", i.e., approximate magnitude

Local acceleration = \( \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \{ U(x) \cos (\omega t - kx - \phi) \} = O(\omega U) \)

Advective acceleration = \( u \frac{\partial u}{\partial x} = u \frac{\partial}{\partial x} \{ U(x) \cos (\omega t - kx - \phi) \} = O( U(dU/dx + kU) ) \)

\( \approx \) \( O( U(U/L_u + kU) ) \) , where \( L_u \) is the length-scale over which \( U \) changes

Pressure gradient = \( g \frac{\partial h}{\partial x} = g \frac{\partial}{\partial x} \{ a(x) \cos (\omega t - kx) \} = O( g(da/dx + ka) ) \)

\( \approx \) \( O( g(a/L_a + ka) ) \) , where \( L_a \) is the length-scale over which \( a \) changes

Bottom friction = \( \tau_a/(\rho h) = rU \cos (\omega t - kx - \phi) \approx O(rU) \)

Bottom friction/Local acceleration = \( rU/(\omega U) = r/\omega \)

Advective acceleration/Local acceleration = \( U(U/L_u + kU)/(\omega U) = U/(\omega L_u) + kU/\omega = U/(\omega L_u) + U/c \)
\[ \frac{b \partial \eta}{\partial t} = -\frac{\partial}{\partial x} (\text{whu}) \]

Rate of change in cross-sectional area of estuary = along-channel convergence in flux in channel
(outside the channel the along-channel velocity is assumed to be negligible)

\[ \Delta b = \frac{b_{\text{HIGH TIDE}} - b_{\text{LOW TIDE}}}{2} \]

\[ \text{Change in x-sect area} = b \frac{\partial \eta}{\partial t} = (\langle b \rangle + (\Delta b/a) \eta) \frac{\partial \eta}{\partial t} = \langle b \rangle \frac{\partial \eta}{\partial t} + (\Delta b/a) \eta \frac{\partial \eta}{\partial t} \]

= Linear + Non-linear term = \( O(b \omega a + (\Delta b/a) a \omega a) \) = \( O(b \omega a + (\Delta b) \omega a) \)

Ratio of non-linear to linear term in change of x-sect area = \( \Delta b/b \)

\[ \frac{b}{\partial t} = -\frac{\partial}{\partial x} (\text{whu}) \]

Rate of change in cross-sectional area of estuary = along-channel convergence in flux in channel

\[ \frac{\partial}{\partial x} (\text{whu}) = w \langle h \rangle u \left[ 1 + \left( \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} + \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) \right) \right] \]

linear part non-linear part

= \( w \langle h \rangle u \{ 1 + (a/\langle h \rangle)(\eta/a) \} (1/L_w + 1/L_h + 1/L_U + k) \)

along-channel change in:
width depth velocity amplitude velocity phase

Ratio of non-linear to linear portion in along-channel flux convergence = \( a/h \)
Ratio of along channel change in velocity phase to change in width = \( kL_w \), etc.
Estimating characteristic length-scales from data:

(i) Estimating wave number, k:

\[ \text{Phase} = kx \]  
(First translate degrees or hours to radians)

In Matlab: \( f = \text{polyfit}(x, \text{phase}, 1); \ k = f(1) \)

Result: 
- Delaware: \( k = \frac{1}{(58 \text{ km})} \pm 2\% \) (one standard error)
- Thames: \( k = \frac{1}{(70 \text{ km})} \pm 18\% \)
- Tamar: \( k = \frac{1}{(57 \text{ km})} \pm 14\% \)

(ii) For other characteristic length scales, fit observations to:

\[ F(x) = F_o \exp(\pm x/L_F) \]  
(such that \( L_F \) is positive)

Then

\[ \log(F) = \log(F_o) \pm \left(\frac{1}{L_F}\right)x \]

In Matlab: \( f = \text{polyfit}(x, \log(F), 1); \ 1/L_F = \text{abs}(f(1)) \)

Figure 3. Estimates of channel cross-sectional area at mid-depth, time-averaged stream width, and cross-sectionally averaged velocity amplitude as a function of distance along the (a) Delaware (\( A \) and \( b \) from Parker [1984], \( U \) from Horner [1995]), (b) Thames (\( A \) from Hunt [1964] and USDMC charts 37145 and 37146, \( b \) from Hawa [1964], \( U \) from Chamber [1956], and (c) Tamar (\( A \), \( b \), and \( U \) from Nuxes et al. [1983]), along with least squares log-linear regressions.

\[
\begin{array}{cccc}
\text{L} & 1/L_A & 1/L_B & 1/L_U \\
\text{Delaware} & 215 \text{ km} & (1/38) \text{ km}^{-1} \pm 3\% & (1/40) \text{ km}^{-1} \pm 3\% & (1/1700) \text{ km}^{-1} \pm 109\% \\
\text{Thames} & 95 \text{ km} & (1/19) \text{ km}^{-1} \pm 4\% & (1/25) \text{ km}^{-1} \pm 3\% & (1/280) \text{ km}^{-1} \pm 39\% \\
\text{Tamar} & 21 \text{ km} & (1/5.3) \text{ km}^{-1} \pm 4\% & (1/4.6) \text{ km}^{-1} \pm 9\% & (1/160) \text{ km}^{-1} \pm 109\% \\
\end{array}
\]
A. Describing the observed tide in estuaries (Lecture 1) pp. 1-4
B. Scaling the dynamics of tides in estuaries (Lecture 1) pp. 5-11
C. Tides in short estuaries (Lecture 2) pp. 12-16
D. Tides in long, shallow, funnel-shaped estuaries (Lecture 2,3) pp. 17-25
E. Tides in long, deep, straight estuaries (Lecture 3) pp. 26-32
F. Tides in long, shallow, straight estuaries (Lecture 4) pp. 33-38
G. Tides in long estuaries, general "equilibrium" estuary (Lecture 4) pp. 39-40
H. Tides in intermediate length, deep, constant width estuaries (Lecture 5) pp. 41-45
I. Tides in intermediate length, constant width, arbitrary depth estuaries (Lecture 5) pp. 46-48
J. Summary (Lecture 5) p. 49

Tides in short estuaries: Using continuity alone to solve for $U$:

$$b \frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial x} (whu)$$

$$\int_{x}^{x=L} b \frac{\partial \eta}{\partial t} \, dx = - \int_{x}^{x=L} \frac{\partial}{\partial x} (whu) \, dx$$

R.H.S. $= (whu \big|_{x=L} - whu \big|_{x}) = whu$

No tidal flux at landward end of estuary

$$L.H.S. = \left[ 1 + O\left( \frac{kL + L/L_{a}}{\alpha} \right) \right] \frac{\partial \eta_{0}}{\partial t} \int_{x}^{x=L} b \, dx$$

Errors due to along-estuary variation in (i) tidal phase (ii) tidal amplitude

$$u(x,t) = \frac{\partial \eta_{0}}{\partial t} \frac{1}{wh} \int_{x}^{x=L} b \, dx = \frac{\partial \eta_{0}}{\partial t} \frac{A_{b}}{A_{c}}$$

with error $O\left( \frac{kL + L/L_{a}}{\alpha} \right)$

$A_{b} = \text{estuarine surface area upstream of } x$

$A_{c} = \text{estuarine channel cross-sectional area at } x$

Small error if estuary is short relative to $1/k$ and $L_{a}$
Error from using continuity alone to solve for $U$:

$$u(x,t) = \frac{\partial \eta_0}{\partial t} \frac{A_b}{A_c}$$

with error $O(kL + L/L_a)$

(Friedrichs & Aubrey, JGR, 99: 3321-3336)

<table>
<thead>
<tr>
<th>System</th>
<th>Length (km)</th>
<th>$L^{-1}$</th>
<th>Error (%)</th>
<th>$kL + L/L_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delaware</td>
<td>215</td>
<td>1/570</td>
<td>±15%</td>
<td>4.0 ± 3%</td>
</tr>
<tr>
<td>Thames</td>
<td>95</td>
<td>1/1500</td>
<td>±270%</td>
<td>1.4 ± 30%</td>
</tr>
<tr>
<td>Tamar</td>
<td>21</td>
<td>1/88</td>
<td>±35%</td>
<td>0.6 ± 22%</td>
</tr>
</tbody>
</table>

None of these systems is “short” because $kL + L/L_a$ are all too big.

“Small” ratio depends on situation, but errors $\leq \sim 1/4$ are often sufficiently small to neglect at lowest order and still gain useful insight into physics.

Using continuity alone to solve for $U$ in very short estuaries: $O(kL + L/L_a) \ll 1$

$$u(x,t) = \frac{\partial \eta_0}{\partial t} \frac{A_b}{A_c} \quad A_b = \text{estuarine surface area upstream of } x \quad A_c = \text{estuarine channel cross-sectional area at } x$$

At lowest order, $\eta_0(t) = a_0 \cos \omega t$, giving

$$u(x,t) = -a_0 \omega (\sin \omega t) \{A_b(x,t)/A_c(x,t)\}$$

Spatial variation in tidal velocity amplitude:

$$U(x) = a_0 \omega A_b(x)/A_c(x)$$
Using continuity alone to solve for $U$ in very short estuaries (cont.): $O(kL + L/L_a) \ll 1$

Spatial variation in tidal velocity amplitude: $U(x) = a_o \omega A_b(x)/A_c(x)$

$p.15$

$u(t)$ in entrance of very short estuary: $u(t) = \frac{\partial \eta_0}{\partial t} A_b = \frac{\partial \eta_0}{\partial t} \left( \frac{A_b(t)}{w(t)} \right)$

Case (i): Estuarine hypsometry dominated by tidal variations in channel depth. Currents in channel strongest closer to low tide when depth is small.

Case (ii): Hypsometry dominated by tidal variations in estuary surface area. Currents in channel strongest closer to high tide when surface area is large.

(Pethick, 1980, ECSS, 11: 331-345)
Tides in long estuaries: (1) Shallow, funnel-shaped estuaries -- lowest order solution

Scaling Continuity

\[ b \frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial x} (whu) \]

Rate of change in cross-sectional area of estuary = along-channel convergence in flux in channel

\[ \Delta b = (b_{\text{HIGH TIDE}} - b_{\text{LOW TIDE}})/2 \]

Change in x-sect area = \( b \frac{\partial \eta}{\partial t} = (\langle b \rangle + (\Delta b/a) \eta) \frac{\partial \eta}{\partial t} = \text{Linear} + \text{Non-linear term} = O(b \omega a + (\Delta b) \omega a) \)

Ratio of non-linear to linear term in change of x-sect area = \( \Delta b/b \)

For first-order case, assume \( \Delta b/b << 1 \) and \( w = b \)

Change in x-sect area:

\[ \langle b \rangle \frac{\partial \eta}{\partial t} \approx w \frac{\partial \eta}{\partial t} \]
(1) Shallow, funnel-shaped estuaries -- scaling continuity (cont.)

\[
\frac{\partial}{\partial x} (\text{whu}) = w \langle h \rangle + \eta \mu \left( \frac{1}{W} \frac{\partial w}{\partial x} + \frac{1}{h} \frac{\partial h}{\partial x} + \frac{1}{U} \frac{\partial u}{\partial x} \right)
\]

linear part \hspace{1cm} \text{non-linear part}

= O( w \langle h \rangle u \{ 1 + (a/\langle h \rangle) \} (1/L_w + 1/L_h + 1/L_U + k) )

along-channel change in:

- width
- depth
- velocity amplitude
- velocity phase

For first-order case, assume $L_w = L_b << L_h, L_U, 1/k$. Also assume $a/h << 1$

For first-order "funnel" shape, $w = b = w_o \exp(-x/L_b)$

\[
(1/w) \frac{\partial w}{\partial x} = \frac{1}{w_o \exp(-x/L_b)} \frac{\partial w_o \exp(-x/L_b)}{\partial x} = \exp(x/L_b) (-1/L_b) \exp(-x/L_b) = -1/L_b
\]

Then $\frac{\partial}{\partial x} (\text{whu}) = w \langle h \rangle u (-1/L_b) = - \text{whu}/L_b$

\[
\text{Continuity:} \quad b \frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial x} (\text{whu}) \quad \rightarrow \quad w \frac{\partial \eta}{\partial t} = \frac{\text{whu}}{L_b} \quad \rightarrow \quad \frac{\partial \eta}{\partial t} = \frac{\text{hu}}{L_b}
\]

(1) Shallow, funnel-shaped estuaries -- continuity (cont.)

\[
\frac{\partial \eta}{\partial t} = \frac{\text{hu}}{L_b}
\]

Change in cross-sectional area = Flux convergence due to width change

Solve for velocity: $u = (L_b/h) \frac{\partial \eta}{\partial t}$

Elevation is “defined” to be: $\eta = a \cos (\omega t - kx)$

Then velocity is: $u = (L_b/h) (-a \omega) \sin (\omega t - kx) = - a \omega L_b/h \sin (\omega t - kx) = - U \sin (\omega t - kx)$

Tidal velocity amplitude: $U = a \omega L_b/h$

Trig identity: $u = - U \sin (\omega t - kx) = U \sin (kx - \omega t) = U \cos (kx - \omega t + \pi/2) = U \cos (\omega t - kx - \pi/2)$

From earlier lecture, $u$ is “defined” to be: $u = U \cos (\omega t - kx + \phi)$

Then velocity relative phase, $\phi = -\pi/2$ = - 90 deg, i.e., Standing Wave relation
(1) Shallow, funnel-shaped estuaries -- Example: Thames River Estuary

Solution: \( \eta = a \cos (\omega t - kx) \), \( u = -U \sin (\omega t - kx) \), \( U = a \omega \frac{L_b}{h} \), \( \phi \approx -90 \text{ deg} \)

Assumes: \( a/h \ll 1 \), \( \Delta b/b \ll 1 \), \( L_w = L_b \ll L_h \), \( L_U \), \( 1/k \)

Values for Thames: \( a/h \approx 0.24 \), \( \Delta b/b \approx 0.17 \),

\[
L_w \approx 18 \text{ km}, \quad L_b \approx 25 \text{ km}, \quad L_h \approx 70 \text{ km}, \quad L_U \approx 280 \text{ km}, \quad 1/k \approx 70 \text{ km}
\]

Poorest approximations are \( L_U/L_h \approx 0.36 \ll 1 \) and \( kL_b \approx 0.36 \ll 1 \)

Solution predicts: Velocity relative phase \( \phi \approx -90 \text{ deg} \)

Velocity amplitude \( U \approx a \omega \frac{L_b}{h} \approx (2.0 \text{ m})(1.4 \times 10^{-4} \text{ s}^{-1})(25 \times 10^3 \text{ m})/(8.5 \text{ m}) = 0.82 \text{ m/s} \)

For example, the length scale for along-channel variation in depth, \( L_h \approx 70 \text{ km} \), is relatively short. Thus we don't expect the analytical predictions to be exact.
Tides in long estuaries: (1) Shallow, funnel-shaped estuaries -- lowest order solution

Scaling Momentum

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial \eta}{\partial x} - ru \]

Local acceleration + advective acceleration = pressure gradient + bottom friction

Order of magnitude: \( \pm O(\omega U) \pm O(U(U/L_U + kU)) \approx \pm O(\omega (a/L_a + ka)) \pm O(rU) \)

Adective acceleration/Local acceleration = \( (U/\omega)(1/L_U + k) \)

\[ = \left\{ \frac{(0.8 \text{ m/s} / 1.4 \times 10^{-4} \text{ s}^{-1})}{1/(280 \text{ km}) + 1/(70 \text{ km})} \right\} = 0.10 \ll 1 \]

Neglecting advective acceleration, momentum becomes:

\[ \frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x} - ru \]

Plug in \( u = -U \sin(\omega t - kx) \), \( \eta = a \cos(\omega t - kx) \) and get:

\[ -\omega \cos(\omega t - kx) = -gak \sin(\omega t - kx) + rU \sin(\omega t - kx) \]

Because phase doesn’t match other terms, nothing can balance this at lowest order. It must be a second order term.

These two terms can cancel at lowest order.

i.e., If convergence in width dominates continuity (\( L_W \approx L_B << L_H, L_U, 1/k \)), it must follow that friction dominates acceleration (\( \omega r << 1 \)) -- we’ll confirm this in a bit.

Reason -- If friction didn’t dominate acceleration, rapid convergence would cause amplitude to increase along channel (creating small \( L_U \)), i.e., the estuary would be “hypersynchronous”.

Key features of the observed tidal velocity are similar to the predicted trend:

1) Observed \( U = O(0.8 \text{ m/s}) \) (within \( \approx 36\% \))

2) Velocity leads tidal elevation with a relative phase of about \(-90 \text{ deg.}\)
Tides in long estuaries: (1) Shallow, funnel-shaped estuaries -- Scaling momentum (cont.)

Lowest-order Momentum: \[ 0 = -g \frac{\partial \eta}{\partial x} - ru \]

\[ 0 \approx -g a k \sin (\omega t - kx) + ru \sin (\omega t - kx) \quad \Rightarrow \quad U = g a k/r \]

From continuity we also had \[ U = a \omega L_b/h \], eliminating \( U \) and \( a \) gives:

\[ \text{Phase speed} \quad c = \frac{\omega}{k} = \frac{gh}{rL_b} \]

Then for Thames (from rearranging relation above relation for \( c \))

\[ \frac{\omega}{r} = \frac{\omega^2 L_b}{(kgh)} = \left(1.4 \times 10^{-4} \text{ s}^{-1}\right)^2(25 \text{ km})/\left((9.8 \text{ m/s}^2)(8.5 \text{ m})(1/70 \text{ km})\right) \approx 0.4 \]

which is consistent with assuming that \( kL_b = 0.36 \ll 1 \)

---

Tides in long estuaries: (1) Shallow, funnel-shaped estuaries -- Gives “1st Order Wave Equation”

Lowest-order Momentum: \[ 0 = -g \frac{\partial \eta}{\partial x} - ru \]

Lowest-order Continuity: \[ \frac{\partial \eta}{\partial t} = \frac{hu}{L_b} \]

Eliminate \( u \) and get: \[ \frac{\partial \eta}{\partial t} + \frac{gh}{rL_b} \frac{\partial \eta}{\partial x} = 0 \]

or, equivalently \[ \frac{\partial \eta}{\partial t} + c \frac{\partial \eta}{\partial x} = 0 \]

In physics, this is known as a “1st-Order Wave Equation” because the derivatives are \( \partial/\partial t \) and \( \partial/\partial x \), not \( \partial^2/\partial t^2 \) and \( \partial^2/\partial x^2 \), which would make it a “2nd-Order Wave Equation”. A 1st-order wave equation allows propagation of waves in only one direction, in this case toward +x.

Reflected waves (which propagate toward -x) are not allowed. Physically, reflected waves don’t occur in this type of system because waves propagating toward -x have their energy spread by divergence of width as well as being damped by friction. Waves propagating toward +x are frictionally damped too, but convergence concentrates wave energy quickly enough to maintain an equilibrium, constant wave amplitude.
Tides in long estuaries: (2) Deep, straight estuaries -- lowest order solution

Scaling Continuity

\[
\frac{b \partial \eta}{\partial t} = - \frac{\partial}{\partial x} \left( whu \right)
\]

For first-order case, assume \( k >> \frac{1}{L_w}, \frac{1}{L_h}, \frac{1}{L_a}, \frac{1}{L_U} \).
Also assume \( a/h << 1, \Delta b/b << 1, w = b \)

L.H.S.: \( b \frac{\partial \eta}{\partial t} = w \frac{\partial}{\partial t} \left( a \cos \left( \omega t - kx \right) \right) = - w a \sin \left( \omega t - kx \right) \)

R.H.S.: \( - \frac{\partial}{\partial x} \left( whu \right) = w h \frac{\partial u}{\partial x} = - w h \frac{\partial}{\partial x} \left( U \cos \left( \omega t - kx + \phi \right) \right) = - w h k U \sin \left( \omega t - kx + \phi \right) \)

L.H.S. = R.H.S. : \(- w o a \sin \left( \omega t - kx \right) = - w h k U \sin \left( \omega t - kx + \phi \right) \)

Then \( U = \frac{a}{h} \frac{\omega}{k} \) \( \phi = 0 \) deg \( \) Progressive wave relation
(2) Long, deep, straight estuaries -- Example: Chesapeake Bay

Solution:

\[ \eta = a \cos(\omega t - kx), \quad u = U \cos(\omega t - kx), \quad \phi = 0 \text{ deg} \]

Chesapeake Bay CBOS Buoy, mid-bay, \( x \approx 170 \text{ km from mouth} \)

![Graph showing along-channel velocity and surface elevation](attachment:image.png)

Date in February 1999

Observed \( \phi \approx 0 \text{ to -30 deg} \), observed \( U \approx 25 \text{ cm/s} \)

Tides in long estuaries: (2) Deep, straight estuaries -- lowest order solution

Scaling Momentum:

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial \eta}{\partial x} - ru \]

Neglect \( u \frac{\partial u}{\partial x} \) relative to \( \frac{\partial u}{\partial t} \), which assumes \( (U/\omega)(1/L_U + k) \ll 1 \)

Neglect \( ru \) relative to \( \frac{\partial u}{\partial t} \), which assumes \( r/\omega \ll 1 \)

Momentum becomes:

\[ \frac{\partial u}{\partial t} \approx -g \frac{\partial \eta}{\partial t} \]

Plug in \( u = U \cos(\omega t - kx), \ \eta = a \cos(\omega t - kx) \) from continuity and get:

\[ -\omega U \sin(\omega t - kx) = -g \frac{ak}{\omega} \sin(\omega t - kx), \quad \text{which gives} \quad U = gak/\omega \]

From continuity we had \( U = (a/h)(\omega/k) \)

Eliminating \( U \) gives:

\[ \omega/k = (gh)^{1/2} \quad U = (a/h)(gh)^{1/2} \]
(2) Long, deep, straight estuaries -- Example: Chesapeake Bay

Solution:

\[ \eta = a \cos(\omega t - kx) , \quad u = U \cos(\omega t - kx) , \quad U = \left( \frac{a}{h} \right) \left( gh \right)^{1/2} , \quad \phi = 0 \text{ deg} , \quad \omega k = \left( gh \right)^{1/2} \]

Assumes: \( 1/k \ll L_w, L_h, L_a, L_U \). Also \( a/h \ll 1, \Delta b/b \ll 1, r/\omega \ll 1 \)

Wave number \( k = (\Delta kx)/(\Delta x) = (1 \text{ radian})(80 \text{ km}) \approx 1/(80 \text{ km}) \)

Tidal amplitude \( a \approx 0.25 \text{ m} \)

Observed tidal elevations

Windmill Point, Virginia \( x \approx 70 \text{ km} \)

Solomons Island, Maryland \( x \approx 150 \text{ km} \)

Date in February 1999

Friction/acceleration \( r/\omega \approx c_d U/(h\omega) \)

\[ \approx (0.003) \left( \frac{0.25 \text{ m/s}}{(12 \text{ m})(1.4 \times 10^{-4} \text{ s}^{-1})} \right) = 0.44 \]

Thus neglecting friction is a relatively poor assumption.
Tidal amplitude \( \approx \) (roughly) constant, suggesting balance between channel convergence increasing amplitude and frictional dissipation decreasing amplitude, i.e., \( kL_\beta \approx \omega/r \)

\[
L_\beta = (\omega/r)(1/k) \approx (0.44)^{-1}(80 \text{ km}) \approx 180 \text{ km}
\]

(Seems reasonable…)

In lower bay, width increases landward, leading to rapid decrease in amplitude with distance into estuary. (Higher amplitude on east side is 2-D Kelvin wave… more on this later.)

(Carter & Pritchard, 1988)

Tides in long estuaries: (2) Deep straight estuaries -- Gives “2nd Order Wave Equation”

Lowest-order Momentum:

\[
\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}
\]

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 \eta}{\partial t^2} = 0
\]

Lowest-order Continuity:

\[
\frac{\partial \eta}{\partial t} = -h \frac{\partial u}{\partial x}
\]

\[
\frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} = 0
\]

 Eliminate \( \frac{\partial^2 u}{\partial t \partial x} \) and get: \( \frac{\partial^2 \eta}{\partial t^2} - gh \frac{\partial^2 \eta}{\partial x^2} = 0 \) or, equivalently

In physics, this is known as a “2nd-Order Wave Equation” because the derivatives are \( \frac{\partial^2}{\partial t^2} \) and \( \frac{\partial^2}{\partial x^2} \). A 2nd-order wave equation allows propagation of waves in both +x and -x. Reflected waves (which propagate toward -x) are allowed. Physically, reflected waves can occur in this type of system because the mild rate of channel width change does not change cause rapid divergence of reflected wave energy. Likewise, weak friction has a mild impact on reflected wave amplitude. (In the Chesapeake Bay, width convergence and friction are still sufficient to damp most of the wave reflected off the head of the bay.)
Tides in long estuaries: (3) Shallow, straight estuaries

Continuity

\[ \frac{\partial \eta}{\partial t} = -wh \frac{\partial u}{\partial x} \]

Assumes \( a/h << 1 \), \( \Delta b/b << 1 \), \( k >> 1/L \)

\( w \approx 1/L \) / \( h \). Does not assume \( k >> 1/L \)

Momentum

\[ 0 = -g \frac{\partial \eta}{\partial x} - ru \]

Assumes \( \omega/r << 1 \)

\[ \frac{\partial}{\partial x} \{ \text{Momentum} \} \] to eliminate \( u \) in continuity gives

\[ \frac{\partial \eta}{\partial t} = \frac{wgh}{br} \frac{\partial^2 \eta}{\partial x^2} \]

Or, equivalently,

\[ \frac{\partial \eta}{\partial t} = D \frac{\partial^2 \eta}{\partial x^2} \]

\[ D = \frac{wgh}{br} \]

In physics, this is known as a “Diffusion Equation”, with \( D = \text{Diffusion Coefficient} \)

Solution

\[ \eta = a_0 \exp(-kx) \cos (\omega t - kx) , \quad u = U_0 \exp(-kx) \cos (\omega t - kx - \pi/4) , \quad \phi = -45 \text{ deg} \]

\[ U_0 = \frac{(a_0/h) (b/w) (\omega D)^{1/2}}{2} \]

Phase speed \( c = \omega/k = (2\omega D)^{1/2} \)
(3) Long, shallow, straight estuaries -- Example: The Fleet, England

(Robinson et al., 1983, ECSS, 16: 651-668)

\[ k = kx/x = (1.7 \text{ rad})/(5 \text{ km}) = 1/2.9 \text{ km} \]

\[ b = 390 \text{ m}, \ w = 130 \text{ m}, \ h = 0.7 \text{ m}, \ a_o \approx 0.7 \text{ m} \]

\[ c = \omega/k = (1.4 \times 10^{-4} \text{ s}^{-1})(2.9 \text{ km}) = 0.4 \text{ m/s} \]

\[ D = c^2/(2\omega) = 570 \text{ m}^2/\text{s}, \ \omega/r = \omega Db/(wh) = 0.04 \]

Thus frictional dominance is a very good assumption.

Is it reasonable to neglect nonlinearity? NO:

\[ a/h = (0.5 \text{ m})/(0.7 \text{ m}) = 0.7 \]

Tidal asymmetry in shallow, straight estuaries:

Tidal phase speed \[ c = \frac{\omega/k}{(2\omega D)^{1/2}} = \left( \frac{2\omega whg}{br} \right)^{1/2} \]

Friction parameter \[ r = \frac{8}{3\pi} \left( \frac{cUm}{h} \right) \rightarrow c = \left( \frac{3\pi \omega wh^2}{4 \left( \frac{cUm}{h} \right)^{1/2}} \right)^{1/2} \]

\[ c \sim \frac{h}{(b^{1/2})} = \frac{\text{channel depth}}{\text{estuary width}^{1/2}} \]

If the channel is much deeper at high tide than at low tide, high tide will propagate into the estuary faster.

If the estuary is much wider at high tide than at low tide, low tide will propagate into the estuary faster.

This results in tidal asymmetries which favor flood or ebb dominance.
Tidal asymmetry in shallow, straight estuaries:

Phase speed \( c \sim \frac{h}{(b^{1/2})} = \frac{\text{channel depth}}{\text{estuary width}^{1/2}} \)

**Case (i)**

\[ \frac{h_{HT} (b_{HT})^{1/2}}{h_{LT} (b_{LT})^{1/2}} > 1 \]

so \( C_{HT} > C_{LT} \)

**Flood Dominance**

**Case (ii)**

\[ \frac{h_{HT} (b_{HT})^{1/2}}{h_{LT} (b_{LT})^{1/2}} < 1 \]

so \( C_{HT} < C_{LT} \)

**Ebb Dominance**

---

Tidal asymmetry in shallow, straight estuaries (cont.):

Tidal Phase Speed \( c \sim (\frac{h^2}{b^{1/2}}) \)

\[ \frac{h^2}{b} = \frac{(\langle h \rangle + \eta)^2}{\langle b \rangle + \Delta b (\eta/a)} = \frac{\langle h^2 / b \rangle}{\langle b \rangle + \Delta b (\eta/a)} \]

\[ = \frac{\langle h/b \rangle}{\langle b \rangle + \Delta b (\eta/a)} = \frac{\langle h/b \rangle}{\langle b \rangle} (1 + 2(a/h)(\eta/a) - (\Delta b/b)(\eta/a)) \]

\[ \gamma = 2(a/h) - (\Delta b/b) \]

"asymmetry parameter"

If \( \gamma > 0 \), tidal changes in depth dominate changes in width

Then high tide propagates faster than low tide \( \rightarrow \) **flood dominance**

If \( \gamma < 0 \), tidal changes in width dominate changes in depth

Then low tide propagates faster than high tide \( \rightarrow \) **ebb dominance**
\[
\text{Intertidal Storage Volume} \quad \frac{\text{Volume in Channels}}{\text{Volume in Channels}} = \frac{2a \Delta b}{hb (1 - \Delta b/b)}
\]

\[\gamma = 0 \text{ when } \Delta b/b = 2a/h, \text{ i.e., when } \frac{\text{Intertidal Storage Volume}}{\text{Volume in Channels}} \approx \frac{4(a/h)^2}{(1 - 2a/h)}\]

\[\gamma = 2(a/h) - (\Delta b/b) \text{ = "asymmetry parameter" is a simpler, more physics-based parameter for distinguishing between flood- vs. ebb-dominant (shallow) tidal estuaries.}\]

**Figure 3.4** Ebb and flood dominance in estuaries related to their volumetric and tidal characteristics. After Friedrichs and Aubrey, 1988.

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**MS 520 Part I: Estuarine Tides -- Carl Friedrichs**

A. Describing the observed tide in estuaries (Lecture 1) pp. 1-4
B. Scaling the dynamics of tides in estuaries (Lecture 1) pp. 5-11
C. Tides in short estuaries (Lecture 2) pp. 12-16
D. Tides in long, shallow, funnel-shaped estuaries (Lecture 2,3) pp. 17-25
E. Tides in long, deep, straight estuaries (Lecture 3) pp. 26-32
F. Tides in long, shallow, straight estuaries (Lecture 4) pp. 33-38
G. Tides in long estuaries, general “equilibrium” estuary (Lecture 4) pp. 39-40
H. Tides in intermediate length, deep, constant width estuaries (Lecture 5) pp. 41-45
I. Tides in intermediate length, constant width, arbitrary depth estuaries (Lecture 5) pp. 46-48
J. Summary (Lecture 5) p. 49
Tides in long estuaries: (4) General "equilibrium" estuary (Ex. James River)

Look for solution of form \( \eta = a \cos(\omega t - kx) \), \( u = U \cos(\omega t - kx + \phi) \)

Assume "equilibrium" channel so that \( a, U \) nearly constant in space

Also assume \( k \) and/or \( 1/L_b \gg 1/L_t, 1/L_a, 1/U \).

Also assume \( a/h \ll 1, \Delta b/b \ll 1, w = b \sim \exp(-x/L_b) \)

Do NOT assume \( k >> 1/L_w, 1/L_b \) and do NOT assume \( r/\omega \ll 1 \)

Continuity: Change in x-sect area = divergence in flux due to width and phase

Momentum: Acceleration = Pressure gradient + Friction

Plug in form for \( \eta \) and \( u \) and after some algebra you find lowest order solution:

<table>
<thead>
<tr>
<th>General &quot;equilibrium&quot; estuary</th>
<th>Shallow, strongly Funnel-shaped</th>
<th>Deep, Constant width</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega/r = O(1), kL_b = O(1) )</td>
<td>( \omega/r &lt;&lt; 1, kL_b &lt;&lt; 1 )</td>
<td>( \omega/r &gt; 1, kL_b &gt;&gt; 1 )</td>
</tr>
</tbody>
</table>

\[
U = \frac{\omega a L_b}{h \left(1 + (kL_b)^2\right)^{1/2}}
\]

\[
\tan \phi = -\frac{1}{(kL_b)}
\]

\[\phi = -90 \, \text{deg}\]

\[\phi = 0 \, \text{deg}\]

Tides in long estuaries: (4) General "equilibrium" estuary (cont.)

Next allow \( \eta \) and \( U \) to change slowly along estuary

So that \( a(x) \sim U(x) \sim \exp(\mu kx), \mu = \text{"amplitude growth factor"}\)

If \( \mu > 0 \), \( a \) and \( U \) increase into estuary ("hyper-synchronous")

If \( \mu < 0 \), \( a \) and \( U \) decrease into estuary ("hypo-synchronous")

Scaling requires \(|\mu| << kL_b, \omega/r\)

"Perturbation" approach similar to analysis of asymmetry factor yields

\[
\mu = \frac{\omega}{r} - kL_b
\]

For pure "equilibrium" estuary (a, U constant), \( \omega/r = kL_b \rightarrow c = \omega/k = rL_b \)

Recall from previous notes:

Shallow, strongly Funnel-shaped

Deep, Constant width

\[
c = gh/(rL_b)
\]

\[
c = (gh)^{1/2}
\]

Relation requires \( c = gh/(rL_b) = rL_b = (gh)^{1/2} \) for all "equilibrium" estuaries!

For \( \mu = 0 \), perturbation solution gives

\[
c = (gh)^{1/2} (\omega/r)/(kL_b)
\]
Tides in intermediate length estuaries: (1) Deep, constant width, constant depth

Governing equation from last lecture:
\[ \frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} = 0 , \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \]

Solution:
\[ \eta = a \cos (\omega t - k(x-L)) + a \cos (\omega t + k(x-L)) \]
\[ u = U \cos (\omega t - k(x-L)) - U \cos (\omega t + k(x-L)) \]

Incident wave + Reflected wave

Apply “boundary condition”: \( \eta(x = 0) = a_o \cos \omega t \)
Substitute in, apply trig identities, get
\[ \omega/k = (gh)^{1/2} \]
\[ U = (a/h)(\omega/k) \]

Above relations predict \( k_{obs} = 0 \), i.e., \( c_{obs} = \omega/k_{obs} \) is predicted to be infinite.

Also \( a_{obs} \) and \( U_{obs} \) change in \( x \), and \( \phi = 0 \) (“standing wave relation”).

The situation where these solutions would apply at lowest order should have
\( (gh)^{1/2}/c_{obs} \ll 1 \) and \( L/L_a = O(1) \). If \( L/L_a \ll 1 \), then use the short estuary solution.

Intermediate length, deep, constant width, constant depth  
Ex. Gulf St. Vincent, Australia

Bowers & Lennon, 1990, ECSS, 30: 17-34

\[ k_{obs} = (kL_{obs}/L) = (0.26 \text{ rad})/(120 \text{ km}) = 1/(460 \text{ km}) \]
\[ (gh)^{1/2}/c_{obs} = ((9.8 \text{ m/s}^2)(25 \text{ m}))^{1/2}/(64 \text{ m/s}) = 0.24 \]
\[ c_{obs} = \omega/k_{obs} = (1.4 \times 10^{-4} \text{ s}^{-1})(460 \text{ km}) = 64 \text{ m/s} \]
\[ L_a/L = \Delta a/a_{obs} = (0.3 \text{ m})/(0.4 \text{ m}) = 0.75 \]
Intermediate length, deep, constant width, constant depth:

**Nodes and Antinodes**

\[ a(x) \sim \cos k(L-x), \; U(x) \sim \sin k(L-x) \]

- There is a maximum or “antinode” in elevation amplitude (and a minimum or “node” in velocity amplitude) at \( k(L-x) = 0 \).
- If the system is long enough, there will be a node in elevation (and an antinode in elevation) at \( k(L-x) = \pi/2 \).

An antinode occurs where the incident and reflected waves reinforce each other.
A node occurs where the incident and reflected waves cancel each other.

**FIGURE 10–18 Tides in Seas, Bays, and Gulf.**

In seas, bays, and gulfs, the forced standing waves have a greater height than those created in lakes. This is because the height of the tide at the open end of the basin must be the same as the open ocean. Therefore, the development of a resonant condition between the free and forced standing waves in such basins produce much greater displacements at the antinodes.

(from Thurman, 1981)

Intermediate length, deep, constant width, constant depth: “Quarter wave resonance”

\[ a \sim U \sim 1/(\cos kL) \]

If \( kL = \pi/2 \), then \( \cos kL = 0 \) and \( a \) and \( U \) become infinitely large. This happens because when \( kL = \pi/2 \), the incident and reflected waves for elevation cancel at the mouth, i.e., the node for elevation occurs at the mouth and the ratio of elevation at the mouth relative to the inner estuary is zero. However, the tide at the estuary mouth is constrained to equal the non-zero ocean tidal amplitude, \( a_o \). Thus the tide within the estuary becomes infinitely large in comparison. Of course the estuarine tidal amplitude never actually becomes infinite, since infinitely strong currents would generate intense bed friction, damping tidal amplitude within the tidal estuary (and violating the assumption that \( r/\omega \ll 1 \)).

This phenomena is termed “quarter wave resonance” because when \( kL = \pi/2 \),
\[ L = (\pi/2)(1/k) = (\pi/2) LTide/(2\pi) = LTide/4 \]

Ex. Bay of Fundy, \( h = 100 \) m. Resonant length = \( T (gh)^{1/2}/4 \)
\[ = (12.4 \times 3600 \text{ sec})/(9.8 \text{ m/s}^2)(100 \text{ m})^{1/2}/4 = 350 \text{ km} \]
Tides in intermediate length estuaries: (2) constant width, constant depth, arbitrary depth

Governing equation becomes:
\[
\frac{\partial^2 \eta}{\partial t^2} + r \frac{\partial \eta}{\partial t} - gh \frac{\partial^2 \eta}{\partial x^2} = 0
\]

Properties of both a second-order wave equation and a diffusion equation. Reflected and incident wave, but with wave decaying as it propagates.

Solution:
\[
\eta = a e^{\mu kx} \cos (\omega t - k(x-L)) + a e^{\mu kx} \cos (\omega t + k(x-L))
\]
\[
n = U e^{\mu kx} \cos (\omega t - k(x-L) + \phi) - U e^{\mu kx} \cos (\omega t + k(x-L) + \phi)
\]

Damped reflection: Incident wave + Reflected wave

In this case \( \mu \) is a positive number and always causes the incident and reflected waves to decay with distance as they propagate. Substitute in, apply b.c.s, get

\[
\mu = (1 + (\omega r)^2)^{-1/2}, \quad \mu = 0 \text{ for frictionless case, } \mu = 1 \text{ for strong friction}
\]

\[
\eta / \eta_0 = (1/2)^{1/2} (\cos 2kx + \cosh 2\mu kx)^{1/2}, \quad U = (a/h)(\omega/k)(1 + \mu^2)^{1/2}, \quad \tan \phi = -\mu
\]

\[
2 gh = (\omega/k)^2 (1 + (\tau/\omega)^2)^{1/2} (1 + (1 + (\tau/\omega)^2)^{1/2})
\]
Damped reflection (cont):

Figure 1. The essential physics of the situation can be visualized thus. A damped wave (a) travelling towards the gulf head meets a damped reflected wave (b) travelling in the opposite direction. Near the reflection point, the two have similar amplitudes and the result is a standing wave. Far from the reflection point, wave (a) is considerably larger than wave (b) and the result tends towards a progressive wave.

(Bowers & Lennon, 1990, ECSS, 30: 17-34)
### SUMMARY OF TIDES IN ESTUARIES

<table>
<thead>
<tr>
<th>CASE</th>
<th>MOMENTUM</th>
<th>Continuity Assumptions</th>
<th>U</th>
<th>φ(deg)</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short (Swash Bay)</td>
<td>None</td>
<td>All terms</td>
<td>$kL, L/L_a &lt;&lt; 1$</td>
<td>$a\omega A_b/A_c$</td>
<td>- 90</td>
</tr>
<tr>
<td>Shallow, Funnel (Thames)</td>
<td>Friction + Press. Grad.</td>
<td>$\partial \eta / \partial t +$</td>
<td>$kL_b &lt;&lt; 1, \omega/\rho &lt;&lt; 1, kL_a &gt;&gt; 1$</td>
<td>$a\omega L_b/h$</td>
<td>- 90</td>
</tr>
<tr>
<td>Deep, Straight, Long (Chesapeake)</td>
<td>Accel. + Press. Grad.</td>
<td>$\partial \eta / \partial t +$</td>
<td>$kL_b &gt;&gt; 1, \omega/\rho &gt;&gt; 1, kL_a &gt;&gt; 1$</td>
<td>$(a/h)(gh)^{1/2}$</td>
<td>0</td>
</tr>
<tr>
<td>Long, Straight, Shallow (Fleet)</td>
<td>Friction + Press. Grad.</td>
<td>$\partial \eta / \partial t +$</td>
<td>$\omega/\rho &lt;&lt; 1, kL_a = O(1)$</td>
<td>$(a_o/h)(b/w)(\omega D) e^{-kx}$</td>
<td>- 45</td>
</tr>
<tr>
<td>General &quot;Equilibrium&quot; Estuary (James)</td>
<td>Accel. + Press. Grad. + Friction</td>
<td>$\partial \eta / \partial t +$</td>
<td>$\omega/\rho = kL_b$</td>
<td>$w aL_b$</td>
<td>0 to $gh/(rL_b)$</td>
</tr>
<tr>
<td>Intermediate length, deep, straight (St. Vincent)</td>
<td>Accel. + Press. Grad.</td>
<td>$\partial \eta / \partial t +$</td>
<td>$L/L_a = O(1)$</td>
<td>$(gh)^{1/2}(a/h) \sin k(L-x)$</td>
<td>- 90</td>
</tr>
<tr>
<td>Intermediate length &amp; depth, straight (Long Island)</td>
<td>Accel. + Press. Grad. + Friction</td>
<td>$\partial \eta / \partial t +$</td>
<td>$\omega/\rho = kL_b$</td>
<td>$w aL_b$</td>
<td>0 to - 90</td>
</tr>
</tbody>
</table>