Answer

The Doppler frequency shift of a single target illuminated by a radar beam is given by

$$f = \frac{2v_r}{\lambda}$$

where v_r is the radial velocity (velocity toward or away from the radar antenna) and λ is the radar wavelength. The volume sampled by a radar pulse typically contains a large population of droplets moving with a wide range of different velocities modulated by turbulent motions within the radar sampling volume. The receiver electronics often used to measure the Doppler frequency shift in a radar measures and records the mean Doppler shift frequency, $\bar{f}$, which is related to the mean radial velocity, $\overline{v_r}$. These means are related through

$$\bar{f} = \frac{2\overline{v_r}}{\lambda}$$

Consider a situation in which the number of droplets, N_{v_r} , with radial velocities falling in a specified narrow range of radial velocities, dN_{v_r} , is given by

$$dN_{v_r} = N_{v_r} exp(-\beta^2 v_r^2) dv_r$$

where dN_{v_r} is an increment in the number of particles with a specified radial velocity and β is a constant. The mean velocity is given by

$$\overline{v_r} = \frac{\int_0^\infty v_r dN_{v_r}}{N}$$

The Root-Mean-Square (or RMS) of a quantity is often measured by devices including a radar receiver that are attempting to characterize a time varying signal. It is given by

$$x_{rms} = \sqrt{\bar{x}^2}$$

Compute $\overline{v_r}$, $\bar{f}$, and f_{rms} as a function of β and λ. Does a measurement of f_{rms} enable a determination of $\overline{v_r}$ if the velocities of the targets are distributed according to the equation above?

$$\overline{v_r} = \frac{\int_0^\infty v_r N_{v_r} exp(-\beta^2 v_r^2) dv_r}{N} = \frac{N_{v_r}}{N} \int_0^\infty v_r exp(-\beta^2 v_r^2) dv_r$$

We must now evaluate the integral, but there is a caveat that we need to consider. The radial velocity v_r can be either toward or away from the radar, which we will consider later. The integral is written as

$$\int_0^\infty v_r exp(-\beta^2 v_r^2) dv_r = \int_0^\infty u\beta^{-2} exp(-u^2) du = \beta^{-2} \int_0^\infty u exp(-u^2) du$$

if $u = \beta v_r$. The definite integral is evaluated as

$$\int_0^\infty u exp(-u^2) du = \frac{1}{2\beta^2}$$

BONUS: If you realize that the integral bounds are from 0 to ∞ and note that this considers only droplets moving in one direction, either toward or away from the radar, you can consider droplets moving in both directions by noting that in the opposite direction the bounds of integration change in the following manner:

$$\int_0^\infty v_r exp(-\beta^2 v_r^2) dv_r + \int_{-\infty}^0 v_r exp(-\beta^2 v_r^2) dv_r = \frac{1}{2\beta^2} - \frac{1}{2\beta^2} = 0$$

This is because we are computing a volume average velocity and in this case there are just as many particles moving toward the radar as away from it at all velocities, so the average velocity is zero.

Back to the question at hand. The average velocity is

$$\overline{v_r} = \frac{N_{v_r}}{N} \frac{1}{2\beta^2} = \frac{1}{2\beta^2}$$

where the last expression on the right arises only if we consider one direction (either toward or away from the radar). The average velocity in the volume is zero if both directions are considered. Let us now compute the Doppler frequency shift and the RMS velocity:

$$\bar{f} = \frac{2\overline{v_r}}{\lambda} = \frac{1}{\lambda\beta^2}$$

$$f_{rms} = \frac{1}{\lambda\beta^2}$$

We can determine that

$$\overline{v_r} = \frac{\lambda f_{rms}}{2}$$

which shows that the average velocity is directly proportional to the RMS frequency shift.